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Internal residual stresses in elastically homogeneous solids: I. Statistically homogeneous stress fluctuations

V.A. Buryachenko*, [1]

Air Force Research Laboratory, Materials Directorate, AFRL/MLBC, Bldg. 654, Wright–Patterson AFB, OH 45433-7750, USA

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Abstract

We consider a linearly elastic composite medium, which consists of a homogeneous matrix containing a homogeneous and statistically uniform random set of ellipsoidal inclusions. The elastic properties of the matrix and the inclusions are the same, but the stress-free strains are different. One obtains the estimation of n-degree moments of stresses averaged over the components. A relation for the statistical stress moments in the matrix in the vicinity of an individual inclusion is also derived. Furthermore one estimates an influence of the fractional composition of the inclusions on the inhomogeneous nature of stress moments inside the inclusions. The expression for the correlation function of stresses is also derived. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In the framework of linear elasticity we consider residual stresses in matrix random structure composites, which arise due to differential or anisotropic eigenstresses (called transformation fields) in the components. These transformation fields result from thermal expansion, phase transformation, twinning and other changes of shape or volume of the material. Knowledge of residual stresses is important e.g. for strength and fracture analyses, or phase transformation predictions. A considerable number of methods are known in the linear theory of such composites which yield the effective elastic constants and stress field averages in the components. Appropriate, but by no means exhaustive, references are provided by the reviews of (Shermergor, 1977; Willis, 1982; Willis, 1983; Kunin, 1983;

* Fax: +001-937-656-7429.

E-mail address: buryach@aol.com (V.A. Buryachenko).

[1] Permanent address: Department of Mathematics, Moscow State University of Engineering Ecology, 107884 Moscow, Russia.

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Mura, 1987; Kreher and Pompe, 1989; Buryachenko and Parton, 1992aBuryachenko and Parton, 1992b; Nemat-Nasser and Hori, 1993). When one tries to estimate the equivalent stress in strength theories as well as in nonlinear creep theory, or when the yield function in plasticity theory is considered, squares of the first or second invariants of the deviator of local stresses are frequently used (see for details and references Buryachenko, 1996). In the extensive recent reviews by Ponte Castañeda (1997) and by Suquet (1997) rigorous variational methods of analyses on different nonlinear problems are presented. The exact relations for all components of the second moment tensor of elastic stresses and internal residual stresses averaged over the volume of the components for anisotropic constituents are obtained by a perturbation method (see references in Buryachenko and Kreher, 1995; Buryachenko, 1996; Buryachenko et al., 1996), based on the assumption that functional dependence of the effective compliance and effective stored energy on the compliance of the components are known. The known disadvantages of this method were eliminated by the method of integral equations by Buryachenko and Rammerstorfer (1997, 1998a). It should be mentioned that an alternative formula for the estimation of the second moment of stresses was proposed by Ju and his co-workers (see e.g. Ju and Chen, 1994; Ju and Tseng, 1994Ju and Tseng, 1996); the disadvantages of their approach were discussed in detail by Buryachenko and Rammerstorfer (1999b).

Especially interesting results are obtained if the fluctuation of elastic compliance are negligible $\mathbf{M}(\mathbf{x})=\text{const}$ whereas the stress-free strains fluctuate. Of course, the first and second moments of residual stresses in composite components can be estimated with the help of the passage to the zero limit of the elastic mismatch of different components in the corresponding formulae by Buryachenko and Kreher (1995). Nevertheless, the desired relationships can be found immediately without some assumptions of the multiparticle effective method (MEFM) proposed by Buryachenko and Kreher (1995). So Kreher and Pompe (1989), Kreher (1990) obtained the exact simple relations for the first stress moments inside a component and for the second moment of stresses averaging over the volume of the composite in which a special case of statistically isotropic materials (no crystallographic texture, no morphological texture) was considered. Kreher and Molinari (1993) generalized this perturbation method to take into account both anisotropic crystal orientation distributions and anisotropic morphology of the microstructure.

In the present paper, a generalization of the method of integral equations by Buryachenko and Rammerstorfer (1997, 1998a) mentioned above is proposed for the estimation of second moments of residual stresses in the components of composites containing a statistically homogeneous field of ellipsoidal uncoated or coated inclusions. Considering both binary and triple interaction of the inclusions explicit relations for second moments of residual stresses are obtained. Under the additional assumptions the statistical moments of residual stresses of any order averaged over a separate phase are presented. The relation for statistical stress moments averaged over the ensemble realization in the matrix in the vicinity of individual inclusions is proposed as well. One estimates an influence of fractional inclusion composition on inhomogeneous nature of stress moments inside inclusions. The derivation of the formulation is expressed in conditional averages of the perturbations which are generated by the surrounding inclusions.

2. Description of the mechanical properties and geometrical structure of components

This paper discusses a certain representative mesodomain w with a characteristic function W containing a set $X=(v_i)$ of inclusions v_i with characteristic functions V_i (i = 1, 2,...). At first no restrictions are imposed on the elastic symmetry of the phases or on the geometry of the inclusions. It is assumed that the inclusions can be grouped into components $v^{(k)}$ (k = 1, 2,..., N) with identical mechanical and geometrical properties. The local strain tensor $\boldsymbol{\epsilon}$ is related to the displacements $\mathbf{u}$ via the linearized strain–displacement equation $\boldsymbol{\epsilon}=[\nabla\otimes\mathbf{u}+(\nabla\otimes\mathbf{u})^{\mathrm{T}}]/2$. Here $\otimes$ denotes tensor product, and $(.)^{\mathrm{T}}$ denotes matrix transposition. The stress tensor $\boldsymbol{\sigma}$, satisfies the equilibrium equation (no body forces acting). Stresses and strains are related to each other via the constitutive equation $\boldsymbol{\sigma}(\mathbf{x})=\mathbf{L}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x})+\boldsymbol{\alpha}(\mathbf{x})$ or $\boldsymbol{\epsilon}(\mathbf{x})=\mathbf{M}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})+\boldsymbol{\beta}(\mathbf{x})$. $\mathbf{L}(\mathbf{x})$ and $\mathbf{M}(\mathbf{x})\equiv\mathbf{L}(\mathbf{x})^{-1}$ are the phase stiffness and compliance fourth-order tensors. $\boldsymbol{\beta}(\mathbf{x})$ and $\boldsymbol{\alpha}(\mathbf{x})\equiv-\mathbf{L}(\mathbf{x})\boldsymbol{\beta}(\mathbf{x})$ are second order tensors of local eigenstrains and eigenstresses (frequently called transformation fields), respectively, which may arise by thermal expansion, phase transformation, twinning and other changes of shape or volume of the material; in particular, thermal strains $\boldsymbol{\beta}=\mathbf{a}^T\theta$, where $\mathbf{a}^T$ is the tensor of linear thermal expansion coefficients and $\theta=T-T_0$ is the temperature change from the reference value T_0 to the current temperature T.

One assumes that the mismatch in the elastic compliance is negligible, i.e. $\mathbf{M}(\mathbf{x})\equiv\mathbf{M}=\text{const.}$, and the stress-free strains $\boldsymbol{\beta}$ fluctuate. The transformation field $\boldsymbol{\beta}$ is decomposed as $\boldsymbol{\beta}\equiv\boldsymbol{\beta}^{(0)}+\boldsymbol{\beta}_1(\mathbf{x})$. $\boldsymbol{\beta}$ is assumed to be constant in the matrix $\boldsymbol{\beta}(\mathbf{x})=\boldsymbol{\beta}^{(0)}$ for $\mathbf{x}\in v^{(0)}=w\backslash v$ ($v\equiv\cup v^{(k)}\equiv\cup v_i$, $k=1,2,\ldots,N$; $i=1,2,\ldots$), and is an inhomogeneous function inside the inclusions: $\boldsymbol{\beta}(\mathbf{x})=\boldsymbol{\beta}^{(0)}+\boldsymbol{\beta}_1^{(k)}(\mathbf{x})$ for $\mathbf{x}\in v^{(k)}\subset v$. Here and in the following, the upper index '(k)' refers to the phases, the lower index i refers to the individual inclusions.

We assume that the phases are perfectly bonded, so that the displacements and the traction components of the stresses are continuous across the interphase boundaries. We take uniform traction boundary conditions for the mesodomain w: $\boldsymbol{\sigma}^0\mathbf{n}(\mathbf{x})=\mathbf{T}(\mathbf{x})$, $\mathbf{x}\in\partial w$, where $\mathbf{T}(\mathbf{x})$ is the traction vector at the external boundary ∂w, $\mathbf{n}$ is its unit outward normal, and where $\boldsymbol{\sigma}^0$ is a given uniform symmetric tensor, representing the macroscopic stress state on the domain w.

It is assumed that the representative mesodomain w contains a statistically large number of inclusions $v_i\subset v^{(k)}$ ($i=1,2,\ldots$; $k=1,2,\ldots,N$); all the random quantities under discussion are described by statistically homogeneous ergodic random fields and, hence, the ensemble averaging could be replaced by volume averaging

$$\langle(.)\rangle=\bar{w}^{-1}\int(.)W(\mathbf{x})\mathrm{d}\mathbf{x}$$

and

$$\langle(.)\rangle^{(k)}=[\bar{v}^{(k)}]^{-1}\int(.)V^{(k)}(\mathbf{x})\mathrm{d}\mathbf{x},\tag{2.1}$$

where $\Sigma V^{(k)}=\Sigma V_i\equiv V$, $k=1,2,\ldots,N$; $i=1,2,\ldots$. $V^{(k)}$ is the characteristic functions of $v^{(k)}$. The bar appearing above the region represents its measure, e.g. $\bar{v}\equiv\text{mes}\,v$. Therefore, the average over component $v^{(k)}$ agrees with the ensemble average over an individual inclusion $v_i\in v^{(k)}$ ($i=1,2,\ldots$): $\langle(.)\rangle_i=\langle(.)\rangle^{(k)}$.

For the description of the random structure of a composite material let us introduce a conditional probability density $\varphi(v_i,\mathbf{x}_i|v_1,\mathbf{x}_1,\ldots,v_n,\mathbf{x}_n)$, which is a probability density to find the i-th inclusion with the center $\mathbf{x}_i$ in the domain v_i with fixed inclusions $v_1,\ldots,v_n$ with the centers $\mathbf{x}_1,\ldots,\mathbf{x}_n$. The notation $\varphi(v_i,\mathbf{x}_i|;v_1,\mathbf{x}_1,\ldots,v_n,\mathbf{x}_n)$ denotes the case $\mathbf{x}_i\neq\mathbf{x}_1,\ldots,\mathbf{x}_n$. Of course, $\varphi(v_i,\mathbf{x}_i|;v_1,\mathbf{x}_1,\ldots,v_n,\mathbf{x}_n)=0$ for values of $\mathbf{x}_i$ lying inside the 'included volumes' $\cup v_{im}^0$ ($m=1,\ldots,n$), where $v_{im}^0\supset v_m$ with characteristic functions V_{0m} (since inclusions cannot overlap), and $\varphi(v_i,\mathbf{x}_i|;v_1,\mathbf{x}_1,\ldots,v_n,\mathbf{x}_n)\to\varphi(v_i,\mathbf{x}_i)$ at $|\mathbf{x}_i-\mathbf{x}_m|\to\infty$, $m=1,\ldots,n$ (since no long-range order is assumed). $\varphi(v_i,\mathbf{x})$ is a number density $n^{(k)}=n^{(k)}(\mathbf{x})$ of component $v^{(k)}\ni v_i$ at the point $\mathbf{x}$ and $c^{(k)}=c^{(k)}(\mathbf{x})$ is the concentration, i.e. volume fraction, of the component $v^{(k)}$ in the point $\mathbf{x}$: $c^{(k)}(\mathbf{x})=\langle V^{(k)}\rangle(\mathbf{x})=\bar{v}_i n^{(k)}(\mathbf{x})$ ($k=1,2,\ldots,N$; $i=1,2,\ldots$), $c^{(0)}(\mathbf{x})=1-\langle V\rangle(\mathbf{x})$. If the pair distribution function $g(\mathbf{x}_i-\mathbf{x}_m)\equiv\varphi(v_i,\mathbf{x}_i|;v_m,\mathbf{x}_m)/n^{(k)}$ depends only on $|\mathbf{x}_m-\mathbf{x}_i|$, it is called the radial distribution function. Hereinafter the notations $\langle(.)\rangle(\mathbf{x})$ and $\langle(.)|v_1,\mathbf{x}_1;\ldots;v_n,\mathbf{x}_n\rangle(\mathbf{x})$ will be used for the average and for the conditional average taken for the ensemble of a statistically inhomogeneous field $X=(v_i)$ at the point $\mathbf{x}$, on the condition that there are inclusions at the points $\mathbf{x}_1,\ldots,\mathbf{x}_n$ and $\mathbf{x}_1\neq\mathbf{x}_n$ for any n. The notation $\langle(.)|;\ v_1,\ \mathbf{x}_1;...;\ v_n,\ \mathbf{x}_n\rangle(\mathbf{y})$ is used for the case $\mathbf{y} \notin v_1,...,\ v_n$. The notation for the conditional probability density $\varphi(v_i,\ \mathbf{x}_i|;\ v_1,\ \mathbf{x}_1,...,\ v_n,\ \mathbf{x}_n;\ \mathbf{x}_0)$ is considered under the condition that the inclusions $v_1,...,\ v_n$ are located in the points $\mathbf{x}_1,...,\ \mathbf{x}_n$, whereas $\mathbf{x}_0$ is the matrix position vector.

3. Average stresses in the components and stored energy

3.1. General relations

It is known that a general integral representation for stresses (see, e.g. Buryachenko and Kreher, 1995)

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}^0 + \int \Gamma(\mathbf{x}-\mathbf{y})[\boldsymbol{\beta}_1(\mathbf{y}) - \langle\boldsymbol{\beta}_1\rangle]\mathrm{d}\mathbf{y}, \tag{3.1}$$

where the integral operator kernel $\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) \equiv -\mathbf{L}^{(0)}[\mathbf{I}\delta(\mathbf{x}-\mathbf{y})+\nabla\nabla\mathbf{G}(\mathbf{x}-\mathbf{y})\mathbf{L}^{(0)}]$, is defined by the infinite-homogeneous-body Green's function $\mathbf{G}$ of the Lamé equation of a homogeneous medium with elastic modulus tensor $\mathbf{L}^{(0)}$, $\mathbf{I}$ is the unit fourth-order tensor.

Let us consider an arbitrary fixed inclusion v_i, then for $\mathbf{x} \in v_i$ from Eq. (3.1), we obtain the relation for the stresses in the inclusion v_1

$$\boldsymbol{\sigma}(\mathbf{x}) = \bar{\boldsymbol{\sigma}}_i(\mathbf{x}) - \mathbf{Q}_i(\mathbf{x},\ \boldsymbol{\beta}_1), \tag{3.2}$$

which is the superposition of the disturbance $-\mathbf{Q}_i(\mathbf{x},\ \boldsymbol{\beta}_1)$ caused by the transformation field in the inclusion considered and the effective field $\bar{\boldsymbol{\sigma}}_i(\mathbf{x})$ produced by the external loading $\boldsymbol{\sigma}^0$ and by the surrounding inhomogeneities:

$$\mathbf{Q}_i(\mathbf{x},\ \boldsymbol{\beta}_1) = -\int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\boldsymbol{\beta}_1(\mathbf{y})V_i(\mathbf{y})\mathrm{d}\mathbf{y},\ \mathbf{x} \in v_i \tag{3.3}$$

and

$$\bar{\boldsymbol{\sigma}}_i(\mathbf{x}) = \boldsymbol{\sigma}^0 + \int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\big[\boldsymbol{\beta}_1(\mathbf{y})(V(\mathbf{y}) - V_i(\mathbf{y})) - \langle\boldsymbol{\beta}_1\rangle\big]\mathrm{d}\mathbf{y}. \tag{3.4}$$

In the case where a single inclusion $v_i \subset v^{(k)}$ is an ellipsoid, then according to the Eshelby theorem, the tensor $\mathbf{Q}_i(\mathbf{x},\ \boldsymbol{\beta}_1)$ has the following properties

$$\mathbf{Q}_i(\mathbf{x},\ \boldsymbol{\beta}_1) = \mathbf{Q}_i\boldsymbol{\beta}^{(k)_1},\ \text{if}\ \boldsymbol{\beta}_1(\mathbf{x}) \equiv \boldsymbol{\beta}^{(k)_1} = \text{const}. \tag{3.5}$$

and

$$\langle\mathbf{Q}_i(\mathbf{x},\ \boldsymbol{\beta}_1)\rangle_i = \mathbf{Q}_i\langle\boldsymbol{\beta}_1\rangle_i, \tag{3.6}$$

where the tensor $\mathbf{Q}_i$ is associated with the well-known Eshelby tensor $\mathbf{S}_i=\mathbf{S}(v_i)$ by $\mathbf{S}_i=\mathbf{I}-\mathbf{M}\mathbf{Q}_i$. Since $\boldsymbol{\beta}_1^{(k)}(\mathbf{x})$ can vanish at the part of the ellipsoidal inclusion $\mathbf{x} \in v_i \subset v^{(k)}$, then an arbitrary nonellipsoidal inclusion v_i can be included into some fictitious ellipsoidal inclusion, and Eq. (3.6) is valid. In the general case for the inclusion v_i the tensor $\mathbf{Q}_i(\mathbf{x},\ \boldsymbol{\beta}_1)$ can be found by numerical methods. For particular cases of coated ellipsoidal inclusions, different analytical models are known (see for references, e.g. Buryachenko and Rammerstorfer, 1996b, 1999a). Averaging Eqs. (3.2) and (3.4) over a random realization of surrounding inclusions, $v_q \neq v_i$ gives

$$\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x}) = \boldsymbol{\sigma}^0 - \mathbf{Q}_i(\mathbf{x},\boldsymbol{\beta}_1) + \int \left[\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\boldsymbol{\beta}_1)\bar{v}_q\varphi(v_q,\mathbf{x}_q|;v_i,\mathbf{x}_i) - \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_q)\langle\boldsymbol{\beta}_1\rangle\right]\mathrm{d}\mathbf{x}_q, \tag{3.7}$$

whence, for ellipsoidal inclusion $\mathbf{x}\in v_i$, it follows that the relation for average stresses in the inclusion v_i is

$$\langle\boldsymbol{\sigma}\rangle_i = \boldsymbol{\sigma}^0 - \mathbf{Q}_i\langle\boldsymbol{\beta}_1\rangle_i + \int \left[\mathbf{T}_{iq}(\mathbf{x}_i-\mathbf{x}_q,\boldsymbol{\beta}_1)\bar{v}_q\varphi(v_q,\mathbf{x}_q|;v_i,\mathbf{x}_i) - \mathbf{T}_i(\mathbf{x}_i-\mathbf{x}_q)\langle\boldsymbol{\beta}_1\rangle\right]\mathrm{d}\mathbf{x}_q, \tag{3.8}$$

hereafter, under $\mathbf{x}\notin v_q$, $v_i\neq v_q$

$$\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\boldsymbol{\beta}_1) = \bar{v}_q^{-1}\int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})V_q(\mathbf{y})\boldsymbol{\beta}_1(\mathbf{y})d\mathbf{y}, \tag{3.9}$$

$$\mathbf{T}_{iq}(\mathbf{x}_i-\mathbf{x}_q,\boldsymbol{\beta}_1) = \left(\bar{v}_i\bar{v}_q\right)^{-1}\int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})V_i(\mathbf{x}_i)V_q(\mathbf{y})\boldsymbol{\beta}_1(\mathbf{y})d\mathbf{x}\,\mathrm{d}\mathbf{y}, \tag{3.10}$$

$$\mathbf{T}_i(\mathbf{x}_i-\mathbf{y}) = \bar{v}_i^{-1}\int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})V_i(\mathbf{x})\mathrm{d}\mathbf{x} \tag{3.11}$$

and

$$\mathbf{T}_{iq}(\mathbf{x}_i-\mathbf{x}_q) = \left(\bar{v}_i\bar{v}_q\right)^{-1}\int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})V_i(\mathbf{x}_i)V_q(\mathbf{y})\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}. \tag{3.12}$$

In Eq. (3.8), $\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})$ stands for average stresses in $\mathbf{x}\in v_i$ over the ensemble realization of surrounding inclusions. Generally speaking, $\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})$ depends on the position $\mathbf{x}$ (contrary to $\langle\boldsymbol{\sigma}\rangle_i$) inside a particular grain or on the orientation of the inclusion itself. Obviously for homogeneous inclusions (i.e. $\boldsymbol{\beta}_1(\mathbf{x})\equiv\boldsymbol{\beta}_1^{(k)}=$const., $\mathbf{x}\in v_q\subset v^{(k)}$, $k=1,\ldots,N$), $\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\boldsymbol{\beta}_1)=\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q)\boldsymbol{\beta}^{(k)}$ and $\mathbf{T}_{iq}(\mathbf{x}_i-\mathbf{x}_q,\boldsymbol{\beta}_1)=\mathbf{T}_{iq}(\mathbf{x}_i-\mathbf{x}_q)\boldsymbol{\beta}^{(k)}$. For an isotropic matrix and spherical inclusions, the tensors $\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q)$ and $\mathbf{T}_{iq}(\mathbf{x}_i-\mathbf{x}_q)$ are known (see, e.g Buryachenko and Rammerstorfer, 1997).

The relation for the stored energy $U^*\equiv-\langle\boldsymbol{\beta}_1\boldsymbol{\sigma}\rangle/2$ (for $\boldsymbol{\sigma}^0\equiv\mathbf{0}$) follows from Eq. (3.8):

$$U^* = -\frac{1}{2}\sum_{i=1}^{N}\Big\{c^{(i)}\langle\mathbf{Q}_i(\mathbf{x},\boldsymbol{\beta}_1)\rangle_i + \int\int V_i(\mathbf{x})\boldsymbol{\beta}_1(\mathbf{x})\left[\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\boldsymbol{\beta}_1)\bar{v}_q\varphi(v_q,\mathbf{x}_q|;v,\mathbf{x}_i)\right. \\ \left. - \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_q)\langle\boldsymbol{\beta}_1\rangle\right]\mathbf{x}_q\,\mathrm{d}\mathbf{x}\Big\}. \tag{3.13}$$

Eqs. (3.7) and (3.13) are new.

3.2. Some particular cases

Eqs. (3.8) and (3.13) may be significantly simplified under the assumption

$$\langle V_q(\mathbf{y})\boldsymbol{\beta}_1^{(q)}(\mathbf{y})|;v_i,\mathbf{x}_i\rangle = \mathbf{f}_1\left(\langle\boldsymbol{\beta}_1^{(q)}\rangle,\rho\right), \tag{3.14}$$

where $\rho\equiv|\mathbf{a}_i^{-1}(\mathbf{x}_q-\mathbf{x}_i)|$. Here the dependence of function $\mathbf{f}_1$ from the geometrical parameters of inclusion v_i is defined by scalar values ρ; $\mathbf{a}_i^{-1}$ identifies a matrix of an affine transformation which transforms the ellipsoid v_i into the unit sphere. According to Eq. (3.14), the conditional averaging properties of the composite have level surfaces which are obtained from the ellipsoidal surfaces by the use of homothetic transformation. Eq. (3.14) holds for some models of composites. In particular, one may consider the grain structure where the correlation of homogeneous transformations $\boldsymbol{\beta}^{(i)}(\mathbf{x}) \equiv \text{const.}$ ($\mathbf{x} \in v^{(i)}$) between different grains is lacking (Kreher and Molinari, 1993). Then Eq. (3.14) is valid under the simplest probability density $\varphi(v_q, \mathbf{x}_q|; v_i, \mathbf{x}_i) = f_2(\rho)$ which is realized for statistical isotropy of a composite structure with spherical inclusions (see Kreher, 1990).

By virtue of the fact that the generalized function, $\boldsymbol{\Gamma}(\mathbf{x})$ is an even homogeneous function of order -3, we have a relation

$$\int [\mathbf{T}(\mathbf{x}-\mathbf{x}_q, \boldsymbol{\beta}_1)\bar{v}_q\varphi(v_q, \mathbf{x}_q|; v_i, \mathbf{x}_i) - \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_q)\langle\boldsymbol{\beta}_1\rangle]\mathrm{d}\mathbf{x}_q = \mathbf{Q}_i\langle\boldsymbol{\beta}_1\rangle \tag{3.15}$$

under the assumption Eq. (3.14). Then Eq. (3.7) can be combined into a simple equation

$$\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x}) = \boldsymbol{\sigma}^0 + \mathbf{Q}_i\langle\boldsymbol{\beta}_1\rangle - \mathbf{Q}_i(\mathbf{x}, \boldsymbol{\beta}_1). \tag{3.16}$$

In so doing for the homogeneous ellipsoidal inclusion v_i, the statistical average stress $\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})$ does not depend on the position $\mathbf{x}$ inside the inclusion being analyzed

$$\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x}) = \boldsymbol{\sigma}^0 + \mathbf{Q}_i\left[\langle\boldsymbol{\beta}_1\rangle - \boldsymbol{\beta}_1^{(i)}\right], \text{ if } \boldsymbol{\beta}_1(\mathbf{x}) = \text{const.}, \mathbf{x} \in v_i. \tag{3.17}$$

In a similar manner, the relation for the stored energy Eq. (3.13) may be simplified as well

$$U^* = \frac{1}{2}\sum_{i=1}^{N} c^{(i)}[\langle\boldsymbol{\beta}_1(\mathbf{x})\mathbf{Q}_i(\mathbf{x}, \boldsymbol{\beta}_1)\rangle_i - \langle\boldsymbol{\beta}_1\rangle_i\mathbf{Q}_i\langle\boldsymbol{\beta}_1\rangle]. \tag{3.18}$$

For homogeneous inclusions, Eq. (3.18) admits of further simplification

$$U^* = \frac{1}{2}\sum_{i=1}^{N} c^{(i)}\boldsymbol{\beta}_1^{(i)}\mathbf{Q}_i\left[\boldsymbol{\beta}_1^{(i)} - \langle\boldsymbol{\beta}_1\rangle\right]. \tag{3.19}$$

Under an additional assumption Eq. (3.14), Eq. (3.19) may be simplified

$$U^* = \frac{1}{2}\langle(\boldsymbol{\beta} - \langle\boldsymbol{\beta}\rangle)\mathbf{Q}(\boldsymbol{\beta} - \langle\boldsymbol{\beta}\rangle)\rangle. \tag{3.20}$$

It should be noted that Eqs. (3.17) and (3.20) for statistically homogeneous and isotropic composites reduce to the results obtained by Kreher (1990). Eq. (3.20) was obtained by Kreher and Molinari (1993) by another less formal method. Kreher (1990) obtained the exact estimation for the second moment of stresses averaged over the whole volume of the composite by the use of a perturbation method

$$\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\rangle = -2\frac{\partial U^*}{\partial\mathbf{M}}\bigg|_{\boldsymbol{\beta}}, \tag{3.21}$$

as well as for the second moment of stresses averaged over the component being analyzed (see also, Buryachenko and Kreher, 1995)

$$\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\rangle_i = -\frac{2}{c^{(i)}}\frac{\partial U^*}{\partial\mathbf{M}^{(i)}}\bigg|_{\boldsymbol{\beta}}, \tag{3.22}$$

where the partial derivative is calculated under a fixed value of stress-free strain $\boldsymbol{\beta}(\mathbf{x})$, $\mathbf{x} \in w$, and the derivative in Eq. (3.22) should be estimated for elastically inhomogeneous medium with the successive passage to the zero limit of the compliance mismatch $\mathbf{M}_1^{(i)} \to \mathbf{0}$. Unfortunately, the estimation of the second moment of stresses inside components Eq. (3.22) is less simple and requires a consideration of multiparticle interactions of inclusions.

3.3. The conditional average of the stresses inside the components

Let $\langle \boldsymbol{\sigma} | v_1, \mathbf{x}_1; v_2, \mathbf{x}_2 \rangle(\mathbf{x})$ denote conditional average stresses over an ensemble realization in the point $\mathbf{x} \in v_1$ under the condition that there are fixed inclusions $v_1 \neq v_2$ in the points $\mathbf{x}_1$ and $\mathbf{x}_2$. This average can be found by the use of the general equation Eq. (3.1) ($\mathbf{x} \in v_1 \neq v_2$)

$$\boldsymbol{\sigma}(\mathbf{x}|v_1, \mathbf{x}_1; v_2, \mathbf{x}_2) = \boldsymbol{\sigma}^0 + \int \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\left[\boldsymbol{\beta}_1(\mathbf{y}) V(\mathbf{y}|v_1, \mathbf{x}_1; v_2, \mathbf{x}_2) - \langle \boldsymbol{\beta}_1 \rangle\right] \mathrm{d}\mathbf{y}, \tag{3.23}$$

where $V(\mathbf{y}|v_1, \mathbf{x}_1; v_2, \mathbf{x}_2)$ is a random characteristic function of inclusions $\mathbf{y}$ under the condition that the inclusions $v_1 \neq v_2$ are located in the domains with the centers $\mathbf{x}_1$ and $\mathbf{x}_2$. The terms with $\mathbf{x} \in v_1$ and $\mathbf{y} \in v_2$ may be isolated in the right-hand-side integral of Eq. (3.23) with the help of the equality $V(\mathbf{y}|v_1, \mathbf{x}_1; v_2, \mathbf{x}_2) = V_1(\mathbf{y}) + V_2(\mathbf{y}) + V(\mathbf{y}|; v_1, \mathbf{x}_1; v_2, \mathbf{x}_2)$ or in terms of the conditional probability density

$$\varphi(v_q, \mathbf{x}_q | v_1, \mathbf{x}_1; v_2, \mathbf{x}_2) = \delta(\mathbf{x}_q - \mathbf{x}_1) + \delta(\mathbf{x}_q - \mathbf{x}_2) + \varphi(v_q, \mathbf{x}_q |; v_1, \mathbf{x}_1; v_2, \mathbf{x}_2). \tag{3.24}$$

Then one averages Eq. (3.23) by the use of Eq. (3.24), ($\mathbf{x} \in v_1$)

$$\begin{aligned} \langle \boldsymbol{\sigma} | v_1, \mathbf{x}_1; v_1, \mathbf{x}_2 \rangle_1(\mathbf{x}) = \boldsymbol{\sigma}^0 - \mathbf{Q}_1(\mathbf{x}, \boldsymbol{\beta}_1) + \mathbf{T}_2(\mathbf{x}, \boldsymbol{\beta}_1) + \int &\left[\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q, \boldsymbol{\beta}_1) \bar{v}_q \varphi(v_q, \mathbf{x}_q |; v_1, \mathbf{x}_1; v_2, \mathbf{x}_2) \right. \\ &\left. - \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_q)\langle \boldsymbol{\beta}_1 \rangle \right] \mathrm{d}\mathbf{x}_q. \end{aligned} \tag{3.25}$$

Taking the relation for average stresses Eq. (3.7) into account, it is possible to rewrite Eq. (3.25) in the form

$$\begin{aligned} \langle \boldsymbol{\sigma} | v_1, \mathbf{x}_1; v_2, \mathbf{x}_2 \rangle(\mathbf{x}) = \langle \boldsymbol{\sigma} \rangle_1(\mathbf{x}) + \mathbf{T}_2(\mathbf{x}, \boldsymbol{\beta}_1) + \int &\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q, \boldsymbol{\beta}_1) \bar{v}_q \left[\varphi(v_q, \mathbf{x}_q |; v_1, \mathbf{x}_1; v_2, \mathbf{x}_2) \right. \\ &\left. - \varphi(v_q, \mathbf{x}_q |; v_1, \mathbf{x}_1)\right] \mathrm{d}\mathbf{y}, \end{aligned} \tag{3.26}$$

where $\mathbf{x} \in v_1$. Proper allowance must be made for calculation of the integral in Eq. (3.26) that the triple conditional probability density $\varphi(v_q, \mathbf{x}_q |; v_1, \mathbf{x}_1; v_2, \mathbf{x}_2)$ is unknown. Note that for $|\mathbf{x}_1 - \mathbf{x}_2| \to \infty$, the assumption

$$\begin{aligned} &\int \left[\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q, \boldsymbol{\beta}_1) \bar{v}_q \varphi(v_q, \mathbf{x}_q |; v_1, \mathbf{x}_1; v_2, \mathbf{x}_2) - \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_q)\langle \boldsymbol{\beta}_1 \rangle \right] \mathrm{d}\mathbf{x}_q \\ &\quad = \int \left[\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q, \boldsymbol{\beta}_1) \bar{v}_q \varphi(v_q, \mathbf{x}_q |; v_1, \mathbf{x}_1) - \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_q)\langle \boldsymbol{\beta}_1 \rangle \right] \mathrm{d}\mathbf{x}_q \end{aligned} \tag{3.27}$$

is equivalent to the acceptability of neglegeble interaction of inclusions. One of the uses of Eq. (3.27) with the assumption Eq. (3.14) leads Eq. (3.26) to a simple representation ($\mathbf{x} \in v_1$) $\langle \boldsymbol{\sigma} | v_1, \mathbf{x}_1; v_2, \mathbf{x}_2 \rangle(\mathbf{x}) = \langle \boldsymbol{\sigma} \rangle_1 + \mathbf{T}_2(\mathbf{x}, \boldsymbol{\beta}_1)$. In such manner, the conditional average stresses $\langle \boldsymbol{\sigma}(\mathbf{x}) | v_1, \mathbf{x}_1; v_2, \mathbf{x}_2 \rangle(\mathbf{x})$ ($\mathbf{x} \in v_i$) is defined by average stresses $\langle \boldsymbol{\sigma} \rangle_1(\mathbf{x})$, Eq. (3.16) and the perturbation $\mathbf{T}_2(\mathbf{x}, \boldsymbol{\beta}_1)$ caused by the inclusion v_2. In a similar manner, the n-point conditional average of stresses in the inclusions $\langle \boldsymbol{\sigma} | v_1, \mathbf{x}_1; v_1, \mathbf{x}_2; \ldots;$

$v_n, \mathbf{x}_n\rangle_1(\mathbf{x})$ ($\mathbf{x} \in v_1$) may be derived under known $(n-1)$-point conditional average ones

$$\langle \boldsymbol{\sigma} | v_1, \mathbf{x}_1; v_2, \mathbf{x}_2; \ldots; v_n, \mathbf{x}_n \rangle(\mathbf{x}) = \langle \boldsymbol{\sigma} | v_1, \mathbf{x}_1; v_2, \mathbf{x}_2; \ldots; v_{n-1}, \mathbf{x}_{n-1} \rangle(\mathbf{x}) + \mathbf{T}_n(\mathbf{x}, \boldsymbol{\beta}_1) + \int \mathbf{T}_q(\mathbf{x} - \mathbf{x}_q, \boldsymbol{\beta}_1) \bar{v}_q [\varphi(v_q, \mathbf{x}_q |; v_1, \mathbf{x}_1; \ldots; v_n, \mathbf{x}_n) - \varphi(v_q, \mathbf{x}_q |; v_1, \mathbf{x}_1; \ldots; v_{n-1}, \mathbf{x}_{n-1})] \mathrm{d}\mathbf{x}_q. \tag{3.28}$$

The right-hand-side integrals in Eqs. (3.26) and (3.28) are of the first order of the inclusion concentration c. Therefore, for dilute concentration of inclusions, their summations are small when compared with $\mathbf{T}_n(\mathbf{x}, \boldsymbol{\beta}_1)$ (n = 2,...).

4. The second moment of stresses inside the components

4.1. Stress fluctuations inside the inclusions

To obtain the second moment of stresses in the component $v^{(i)}$ of the inclusions (i = 1, 2,....), it is necessary to take the tensor product of Eq. 3.1 into $\boldsymbol{\sigma}(\mathbf{x})$, $\mathbf{x} \in v_i$

$$\boldsymbol{\sigma}(\mathbf{x}) \otimes \boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}^0 \otimes \boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^0 \otimes \int \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{z})[\boldsymbol{\beta}_1(\mathbf{z}) V_q(\mathbf{z}) - \langle \boldsymbol{\beta}_1 \rangle] \mathrm{d}\mathbf{z} + \int \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y})[\boldsymbol{\beta}_1(\mathbf{y}) V_p(\mathbf{y}) - \langle \boldsymbol{\beta}_1 \rangle] \mathrm{d}\mathbf{y} \otimes \boldsymbol{\sigma}^0 + \int \int \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y})[\boldsymbol{\beta}_1(\mathbf{y}) V_p(\mathbf{y}) - \langle \boldsymbol{\beta}_1 \rangle] \otimes \Gamma(\mathbf{x} - \mathbf{z})[\boldsymbol{\beta}_1(\mathbf{z}) V_q(\mathbf{z}) - \langle \boldsymbol{\beta}_1 \rangle] \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z}. \tag{4.1}$$

The right-hand-side of Eq. (4.1) is a random function of arrangements of surrounding inclusions v_p, v_q (p, q = 1,2,...). One averages Eq. (4.1) over realization ensemble

$$\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle_i(\mathbf{x}) = \boldsymbol{\sigma}^0 \otimes \boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^0 \int [\mathbf{T}_q(\mathbf{x} - \mathbf{x}_q, \boldsymbol{\beta}_1) \varphi(v_q, \mathbf{x}_q | v_i, \mathbf{x}_i) - \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}_q) \langle \boldsymbol{\beta}_1 \rangle] \mathrm{d}\mathbf{x}_q + \int [\mathbf{T}_p(\mathbf{x} - \mathbf{x}_p, \boldsymbol{\beta}_1) \varphi(v_p, \mathbf{x}_p | v_i, \mathbf{x}_i) - \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}_p) \langle \boldsymbol{\beta}_1 \rangle] \mathrm{d}\mathbf{x}_p + \int \int \big\{ \mathbf{T}_p(\mathbf{x} - \mathbf{x}_p, \boldsymbol{\beta}_1) \otimes \mathbf{T}_q(\mathbf{x} - \mathbf{x}_q, \boldsymbol{\beta}_1) \times \varphi(v_p, \mathbf{x}_p, v_q, \mathbf{x}_q | v_i, \mathbf{x}_i) - \mathbf{T}_p(\mathbf{x} - \mathbf{x}_p, \boldsymbol{\beta}_1) \otimes \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}_q) \langle \boldsymbol{\beta}_1 \rangle \varphi(v_p, \mathbf{x}_p tt | v_i, \mathbf{x}_i) - [\boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}_p) \langle \boldsymbol{\beta}_1 \rangle] \otimes \mathbf{T}_q(\mathbf{x} - \mathbf{x}_q, \boldsymbol{\beta}_1) \varphi(v_q, \mathbf{x}_q | v_i, \mathbf{x}_i) + [\boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}_p) \langle \boldsymbol{\beta}_1 \rangle] \otimes [\boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}_q) \langle \boldsymbol{\beta}_1 \rangle] \big\} \mathrm{d}\mathbf{x}_p\, \mathrm{d}\mathbf{x}_q. \tag{4.2}$$

The right-hand-side of Eq. (4.2) includes one-point and double point conditional probability densities, in which the terms with $\mathbf{x}_p = \mathbf{x}_i$, $\mathbf{x}_q = \mathbf{x}_i$ and $\mathbf{x}_p = \mathbf{x}_q$ may be isolated with the help of the equalities

$$\varphi(v_p, \mathbf{x}_p | v_1, \mathbf{x}_i) = \delta(\mathbf{x}_p - \mathbf{x}_i) + \varphi(v_p, \mathbf{x}_p |; v_i, \mathbf{x}_i),$$

$$\varphi(v_q, \mathbf{x}_q v_i, \mathbf{x}_i) = \delta(\mathbf{x}_q - \mathbf{x}_i) + \varphi(v_q, \mathbf{x}_q |; v_i, \mathbf{x}_i),$$

and

$$\varphi(v_p, \mathbf{x}_p, v_q, \mathbf{x}_q | v_i, \mathbf{x}_i) = \delta(\mathbf{x}_p - \mathbf{x}_q) \delta(\mathbf{x}_q - \mathbf{x}_i) + \delta(\mathbf{x}_p - \mathbf{x}_q) \varphi(v_p, \mathbf{x}_p |; v_i, \mathbf{x}_i) + \delta(\mathbf{x}_p - \mathbf{x}_i) \varphi(v_q, \mathbf{x}_q |; v_i, \mathbf{x}_i) + \delta(\mathbf{x}_q - \mathbf{x}_i) \varphi(v_p, \mathbf{x}_p |; v_i, \mathbf{x}_i) + \varphi(v_p, \mathbf{x}_p |; v_i, \mathbf{x}_i) \varphi(v_q, \mathbf{x}_q |; v_p, \mathbf{x}_p; v_i, \mathbf{x}_i). \tag{4.3}$$

Then Eq. (4.2) may be rewritten as ($\mathbf{x} \in v_i$)

$$\begin{aligned}\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\rangle_i(\mathbf{x}) &= -\boldsymbol{\sigma}^0\otimes\boldsymbol{\sigma}^0+\boldsymbol{\sigma}^0\otimes\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})+\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})\otimes\boldsymbol{\sigma}^0-\mathbf{Q}_i(\mathbf{x},\,\boldsymbol{\beta}_1)\otimes\mathbf{Q}_i(\mathbf{x},\,\boldsymbol{\beta}_1)-\mathbf{Q}_i(\mathbf{x},\,\boldsymbol{\beta}_1)\\ &\otimes[\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})-\boldsymbol{\sigma}^0]-[\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})-\boldsymbol{\sigma}^0]\otimes\mathbf{Q}_i(\mathbf{x},\,\boldsymbol{\beta}_1)+\int\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\otimes\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\\ &\times\bar{v}_p\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i)d\mathbf{x}_p+\int\int\{\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\otimes\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\,\boldsymbol{\beta}_1)\bar{v}_q\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i)\\ &\times\varphi(v_q,\,\mathbf{x}_q|;\,v_p,\,\mathbf{x}_p;\,v_i,\,\mathbf{x}_i)-[\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_p)\langle\boldsymbol{\beta}_1\rangle]\otimes\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\,\boldsymbol{\beta}_1)\bar{v}_q\varphi(v_q,\,\mathbf{x}_q|;\,v_i,\,\mathbf{x}_i)\\ &-[\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i)-\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_p)\langle\boldsymbol{\beta}_1\rangle]\otimes\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_q)\langle\boldsymbol{\beta}_1\rangle\}\mathrm{d}\mathbf{x}_p\,\mathrm{d}\mathbf{x}_q.\end{aligned}\tag{4.4}$$

As this takes place, the last two items in the double right-hand-side integral can be expressed in terms of averaging stresses Eq. (3.7). Then, Eq. (4.4) may be simplified to

$$\begin{aligned}\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\rangle_i(\mathbf{x}) &= \langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})\otimes\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})+\int\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\otimes\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i)d\mathbf{x}_p\\ &+\int\int\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\otimes\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\,\boldsymbol{\beta}_1)\bar{v}_q\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i)[\varphi(v_q,\,\mathbf{x}_q|;\,v_p,\,\mathbf{x}_p;\,v_i,\,\mathbf{x}_i)\\ &-\varphi(v_q,\,\mathbf{x}_q|;\,v_i,\,\mathbf{x}_i)]\mathrm{d}\mathbf{x}_q\,\mathrm{d}\mathbf{x}_p.\end{aligned}\tag{4.5}$$

The new exact relation Eq. (4.5) is derived by the use of triple interaction of the inclusions. As may be seen from Eq. (4.5), neglect of binary interaction is tantamount to the assuming of homogeneity of stresses inside component v_i

$$\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\rangle_i(\mathbf{x})=\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})\otimes\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x}).\tag{4.6}$$

The following approximation of second moment estimation can be obtained by taking into account only the binary interactions of inclusions

$$\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\rangle_i(\mathbf{x})=\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})\otimes\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})+\int\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\otimes\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i)\mathrm{d}\mathbf{x}.\tag{4.7}$$

In Section 7, it will be shown that Eq. (4.7) provides a sufficiently good approximation of the exact solution. One can see from Eq. (4.7) that the covariance matrix

$$\Delta\boldsymbol{\sigma}_i^2(\mathbf{x})\equiv\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\rangle_i(\mathbf{x})-\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})\otimes\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})\tag{4.8}$$

presents a determined nonhomogeneous function of coordinate $\mathbf{x}$ inside inclusion v_i, in contrast to $\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})=\text{const}$. Eq. (4.8) makes possible the estimation of average fluctuations over the inclusion volume

$$\langle\Delta\boldsymbol{\sigma}^2\rangle_i=\bar{v}_i^{-1}\int\Delta\boldsymbol{\sigma}_i^2(\mathbf{x})V_i(\mathbf{x})\mathrm{d}\mathbf{x}.\tag{4.9}$$

It is significant that in deriving Eq. (4.5), we did not use hypothesis H_1 of the so-called MEFM (see Buryachenko and Kreher, 1995 for details):

$$\bar{\boldsymbol{\sigma}}_i(\mathbf{x})=\text{const.},\ \mathbf{x}\in v_i.\tag{4.10}$$

Employing this hypothesis, we obtain a uniform second stress moment inside the inclusion v_i

$$\Delta\boldsymbol{\sigma}_i^{2\mathrm{EFM}} = \int\left[\mathbf{T}_{ip}(\mathbf{x}_i-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\otimes\mathbf{T}_{ip}(\mathbf{x}_i-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\right]\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i)\mathrm{d}\mathbf{x}_p + \int\int\left[\mathbf{T}_{ip}(\mathbf{x}_i-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\right.$$
$$\left.\times\,\bar{v}_p\otimes\mathbf{T}_{iq}(\mathbf{x}_i-\mathbf{x}_q,\,\boldsymbol{\beta}_1)\bar{v}_q\right]\cdot\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i)[\varphi(v_q,\,\mathbf{x}_q|;\,v_p,\,\mathbf{x}_p;\,v_i,\,\mathbf{x}_i)$$
$$-\,\varphi(v_q,\,\mathbf{x}_q|;\,v_i,\,\mathbf{x}_i)]\mathrm{d}\mathbf{x}_p\,\mathrm{d}\mathbf{x}_q. \tag{4.11}$$

In view of problem nonlinearity, $\langle\Delta\boldsymbol{\sigma}^2\rangle_i\neq\Delta\boldsymbol{\sigma}_i^{2\mathrm{EFM}}$. For homogeneous ellipsoidal inclusions, neglecting the second integral in the right-hand-side of Eq. (4.11) reduces Eq. (4.11) to the relation proposed by Buryachenko and Rammerstorfer (1996a) without any justification.

4.2. Conditional second moments of stresses inside the inclusions

The previous method of the estimation of stress second moment can be used for the calculation of conditional moments of stresses. But it is more convenient to directly employ a conditional average Eq. (3.28) for this purpose. At first, we have from general equation Eq. (3.1) for fixed inclusions, $v_i\neq v_j$ ($\mathbf{x}\in v_i$)

$$\boldsymbol{\sigma}(\mathbf{x})\otimes\boldsymbol{\sigma}(\mathbf{x}) = \left\{\boldsymbol{\sigma}^0-\mathbf{Q}_i(\mathbf{x},\,\boldsymbol{\beta}_1)+\mathbf{T}_j(\mathbf{x}-\mathbf{x}_j,\,\boldsymbol{\beta}_1)\bar{v}_j+\int\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})[\boldsymbol{\beta}_1(\mathbf{y})V(\mathbf{y}|;\,v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j)\right.$$
$$\left.-\,\langle\boldsymbol{\beta}_1\rangle]\mathrm{d}\mathbf{y}\right\}\otimes\boldsymbol{\sigma}(\mathbf{x}). \tag{4.12}$$

Taking conditional averaging of Eq. (4.12), we obtain ($\mathbf{x}\in v_i\neq v_j$)

$$\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}|v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j\rangle(\mathbf{x}) = \left[\boldsymbol{\sigma}^0-\mathbf{Q}_i(\mathbf{x},\,\boldsymbol{\beta}_1)+\mathbf{T}_j(\mathbf{x}-\mathbf{x}_j,\,\boldsymbol{\beta}_1)\bar{v}_j\right]\otimes\langle\boldsymbol{\sigma}|v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j\rangle(\mathbf{x})$$
$$+\int\left\{\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\otimes\langle\boldsymbol{\sigma}|v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j;\,v_p,\,\mathbf{x}_p\rangle(\mathbf{x})\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j)\right.$$
$$\left.-\,[\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{x}_p)\langle\boldsymbol{\beta}_1\rangle]\otimes\langle\boldsymbol{\sigma}|v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j\rangle(\mathbf{x})\right\}\mathrm{d}\mathbf{x}_p. \tag{4.13}$$

Transforming Eq. (4.13) by the use of Eqs. (3.26) and (3.28), in which $n=3$, leads to

$$\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}|v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j\rangle(\mathbf{x}) = \langle\boldsymbol{\sigma}|v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j\rangle(\mathbf{x})\otimes\langle\boldsymbol{\sigma}|v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j\rangle(\mathbf{x})+\int\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p$$
$$\otimes\,\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j)\mathrm{d}\mathbf{x}_p+\int\int\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\,\boldsymbol{\beta}_1)\bar{v}_p\otimes\mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\,\boldsymbol{\beta}_1)$$
$$\times\,\bar{v}_q\varphi(v_p,\,\mathbf{x}_p|;\,v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j)\cdot[\varphi(v_q,\,\mathbf{x}_q|;\,v_p,\,\mathbf{x}_p;\,v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j)-\varphi(v_q,\,\mathbf{x}_q|;\,v_i,\,\mathbf{x}_i;\,v_j,\,\mathbf{x}_j)]\mathrm{d}\mathbf{x}_q\,\mathrm{d}\mathbf{x}_p. \tag{4.14}$$

4.3. Stress fluctuations inside the matrix

In a similar spirit, it is possible to obtain the estimation of the second stress moment averaging over a volume of the matrix. If the foregoing reasonings are repeated, we obtain

$$\langle \boldsymbol{\sigma} \otimes \otimes \boldsymbol{\sigma} \rangle_0 = \langle \boldsymbol{\sigma} \rangle_0 \otimes \langle \boldsymbol{\sigma} \rangle_0 + \int \mathbf{T}_p(\mathbf{x}_0 - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p \otimes \mathbf{T}_p(\mathbf{x}_0 - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p \varphi(v_p, \mathbf{x}_p|; v_0, \mathbf{x}_0)\mathrm{d}\mathbf{x}_p$$
$$+ \int\int \mathbf{T}_p(\mathbf{x}_0 - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p \otimes \mathbf{T}_q(\mathbf{x}_0 - \mathbf{x}_q, \boldsymbol{\beta}_1)\bar{v}_q \cdot \varphi(v_p, \mathbf{x}_p|; v_0, \mathbf{x}_0)[\varphi(v_q, \mathbf{x}_q|; v_p, \mathbf{x}_p; v_0, \mathbf{x}_0)$$
$$- \varphi(v_q, \mathbf{x}_q|; v_0, \mathbf{x}_0)]\mathrm{d}\mathbf{x}_p \, \mathrm{d}\mathbf{x}_q. \tag{4.15}$$

The second stress moment inside Eq. (4.15) does not depend on $\mathbf{x}_0 \in v_0$ in contrast to Eq. (4.5).

An approximate estimation of the second moment $\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle_0$ and the conditional one $\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} |; v_i, \mathbf{x}_i \rangle_0$ can be obtained in perfect analogy to Eqs. (4.7) and (4.14) by the use of the replacement of values v_i, $\mathbf{x}_i$ and $\mathbf{x}$ by v_0, $\mathbf{x}_0$ and $\mathbf{x}_0$.

4.4. Stress second moment in composite

The resultant evaluations of stress moments in individual components (Eqs. (4.5) and (4.15)) enable one to estimate the average over the whole of volume of composite. For example, we can obtain an estimation of the second stress moment in the composite from Eqs. (4.5) and (4.15)

$$\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle = \sum_{j=0}^{N} c^{(j)} \langle \boldsymbol{\sigma} \rangle_j \otimes \langle \boldsymbol{\sigma} \rangle_j + \sum_{i=1}^{N} n^{(i)} \int\int [\mathbf{T}_p(\mathbf{x} - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p \otimes \mathbf{T}_p(\mathbf{x} - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p \varphi(v_p, \mathbf{x}_p|; v_i,$$
$$\mathbf{x}_i) V_i(\mathbf{x})\mathrm{d}\mathbf{x}_p \, \mathrm{d}\mathbf{x} + \sum_{i=1}^{N} n^{(i)} \int\int\int \mathbf{T}_p(\mathbf{x} - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p \otimes \mathbf{T}_q(\mathbf{x} - \mathbf{x}_q, \boldsymbol{\beta}_1)\bar{v}_q \varphi(v_p, \mathbf{x}_p|; v_i, \mathbf{x}_i)$$
$$\times [\varphi(v_q, \mathbf{x}_q|; v_p, \mathbf{x}_p; v_i, \mathbf{x}_i) - \varphi(v_q, \mathbf{x}_q|; v_i, \mathbf{x}_i)] V_i(\mathbf{x})\mathrm{d}\mathbf{x}_q \, \mathrm{d}\mathbf{x}_p \, \mathrm{d}\mathbf{x} + c^{(0)} \int \mathbf{T}_p(\mathbf{x}_0 - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p]$$
$$\otimes \mathbf{T}_p(\mathbf{x}_0 - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p \varphi(v_p, \mathbf{x}_p|; v_0, \mathbf{x}_0)\mathrm{d}\mathbf{x}_p + c^{(0)} \int\int \mathbf{T}_p(\mathbf{x}_0 - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p \otimes \mathbf{T}_q(\mathbf{x}_0 - \mathbf{x}_q, \boldsymbol{\beta}_1)$$
$$\times \bar{v}_q \cdot \varphi(v_p, \mathbf{x}_p|; v_0, \mathbf{x}_0)[\varphi(v_q, \mathbf{x}_q|; v_p, \mathbf{x}_p; v_0, \mathbf{x}_0) - \varphi(v_q, \mathbf{x}_q|; v_0, \mathbf{x}_0)]\mathrm{d}\mathbf{x}_q \, \mathrm{d}\mathbf{x}_p. \tag{4.16}$$

Therefore, the exact estimation of $\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle$ in Eqs. (3.20) and (3.21) offers a test of the accuracy of Eq. (4.16), which uses dissimilar approximate conditional probability densities. It should be noted, in connection with this, that the second stress moment in the composite, $\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle$, can be represented in the form

$$\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle = \sum_{j=0}^{N} c_j \langle \boldsymbol{\sigma} \rangle_j \otimes \langle \boldsymbol{\sigma} \rangle_j + \bar{\Delta}\boldsymbol{\sigma}^2, \tag{4.17}$$

where the first term in the right-hand-side of Eq. (4.17) is defined by the average stresses inside the components and the second one, $\bar{\Delta}\boldsymbol{\sigma}^2$, depends on the stress fluctuation inside each component; because $\langle \boldsymbol{\sigma} \rangle \equiv \boldsymbol{\sigma}^0$, the second moment of stresses Eq. (4.17) is, in fact, the stress fluctuations in the composite $\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle \equiv \Delta\boldsymbol{\sigma}^2$ at $\boldsymbol{\sigma}^0 = \mathbf{0}$. By virtue of the fact that the average stresses in the components are calculated exactly by Eq. (3.16), the difference,

$$\bar{\Delta}\boldsymbol{\sigma}^2 = \langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle - \sum_{j=0}^{N} c_j \langle \boldsymbol{\sigma} \rangle_j \otimes \langle \boldsymbol{\sigma} \rangle_j, \tag{4.18}$$

can be used for the estimation of the accuracy of calculated stress fluctuations inside the components (Eqs. (4.5) and (4.15)).

It is pertinent to note that in a number of example cases, simplified methods (like Mori–Tanaka–Eshelby) permit one to obtain a reasonable accuracy for the effective parameters, $\mathbf{M}^*$, $\boldsymbol{\beta}^*$ and U^*. But, for the estimation of central stress moments inside component inclusions, one obtains a trivial result, $\Delta\boldsymbol{\sigma}_i^2(\mathbf{x}) \equiv 0$, $\mathbf{x} \in v^{(i)}$ $(i = 0, 1, \ldots)$.

5. Statistical moments of stresses of arbitrary degree

5.1. Stress moments inside components

The estimation method of second stress moments (Eqs. (4.5) and (4.15)) and the conditional one Eq. (4.14) can be extended to the evaluation of stress moments of any degree. Namely, we take the tensor product of n-degree, according to the general formula (3.1) ($\mathbf{x} \in v_i$)

$$\boldsymbol{\sigma}(\mathbf{x})[\otimes\boldsymbol{\sigma}(\mathbf{x})]^{n-1} = \boldsymbol{\sigma}^0[\otimes\boldsymbol{\sigma}(\mathbf{x})]^{n-1} - \mathbf{Q}_i(\mathbf{x}, \boldsymbol{\beta}_1)[\otimes\boldsymbol{\sigma}(\mathbf{x})]^{n-1} + \int [\Gamma(\mathbf{x}-\mathbf{y})\big[\boldsymbol{\beta}_1(\mathbf{y})V(\mathbf{y}|;\, v_i, \mathbf{x}_i) - \langle\boldsymbol{\beta}_1\rangle\big]\big[\otimes\boldsymbol{\sigma}(\mathbf{x})\big]^{n-1}\mathrm{d}\mathbf{x}, \tag{5.1}$$

where $\boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{n-1} \equiv \boldsymbol{\sigma}\otimes \ldots \otimes\boldsymbol{\sigma}$. Taking the average of Eq. (5.1) over the ensemble realization and making use of Eq. (3.7) relating the mean stress field to the disturbance caused by the surrounding inclusions gives

$$\langle\boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{n-1}\rangle_i(\mathbf{x}) = \langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})\langle(\otimes\boldsymbol{\sigma})^{n-1}\rangle_i(\mathbf{x}) + \int \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p\varphi(v_p, \mathbf{x}_p|;\, v, \mathbf{x}_i) \times \Big[\langle(\otimes\boldsymbol{\sigma})^{n-1}|v_i, \mathbf{x}_i;\, v_p, \mathbf{x}_i\rangle(\mathbf{x}) - \langle(\otimes\boldsymbol{\sigma})^{n-1}\rangle_i(\mathbf{x})\Big]\mathrm{d}\mathbf{x}_p. \tag{5.2}$$

Thus, the estimation problem of the one-point stress moment of order n is reduced to constructing the stress moment and conditional one of order $n-1$. For evaluation of stress moments inside the matrix $\langle\boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{n-1}\rangle_0(\mathbf{x}_0)$, ($\mathbf{x}_0 \in v^0$), it is necessary to replace in Eq. (5.2) the values v_i, $\mathbf{x}_i$ and $\mathbf{x}$ by v_0, $\mathbf{x}_0$ and $\mathbf{x}_0$; in so doing, the stress moments, $\langle\boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{n-1}\rangle_0(\mathbf{x}_0)$ do not depend on the location of $\mathbf{x}_0$ inside the matrix.

To obtain the conditional stress moment of order n under the condition that the inclusion $v_i \neq v_j$ is fixed, let us take a tensor product of values of the field $\boldsymbol{\sigma}(\mathbf{x}|v_i, \mathbf{x}_i;\, v_j, \mathbf{x}_j)$ Eq. (3.23) over the stress moment $(\otimes\boldsymbol{\sigma})^{n-1}$ and average the result, taking Eq. (3.26) into account. Then at $\mathbf{x} \in v_i$, we obtain

$$\langle\boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{n-1}|v_i, \mathbf{x}_i;\, v_j, \mathbf{x}_j\rangle_i(\mathbf{x}) = \langle\boldsymbol{\sigma}|v_i, \mathbf{x}_i;\, v_j, \mathbf{x}_j\rangle_i(\mathbf{x})\langle(\otimes\boldsymbol{\sigma})^{n-1}|v_i, \mathbf{x}_i;\, v_j, \mathbf{x}_j\rangle(\mathbf{x}) + \int \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p, \boldsymbol{\beta}_1) \times \bar{v}_p\varphi(v_p, \mathbf{x}_p|;\, v_i,\mathbf{x}_i;\, v_j, \mathbf{x}_j)\Big[\langle(\otimes\boldsymbol{\sigma})^{n-1}|v_i, \mathbf{x}_i;\, v_p, \mathbf{x}_p;\, v_j, \mathbf{x}_j\rangle_i(\mathbf{x}) - \langle(\otimes\boldsymbol{\sigma})^{n-1}|v_i, \mathbf{x}_i;\, v_j, \mathbf{x}_j\rangle_i(\mathbf{x})\Big]\mathrm{d}\mathbf{x}_p. \tag{5.3}$$

Let us show the calculation $\langle\boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{n-1}\rangle_i(\mathbf{x})$, ($\mathbf{x} \in v_i$) as an example of the estimation of stress moments of the third order. In line with Eq. (5.2), we have

$$\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle_i(\mathbf{x}) = \langle \boldsymbol{\sigma} \rangle_i(\mathbf{x}) \otimes \langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle_i(\mathbf{x}) + \int [\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\, \boldsymbol{\beta}_1)\bar{v}_p \varphi(v_p,\, \mathbf{x}_p|;\, v_i,\, \mathbf{x}_i)$$
$$\times [\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} | v_i,\, \mathbf{x}_i;\, v_p,\, \mathbf{x}_p\rangle(\mathbf{x}) - \langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle_i(\mathbf{x})]\mathrm{d}\mathbf{x}_p, \tag{5.4}$$

According to Eqs. (4.5) and (4.14), the moments $\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle_i(\mathbf{x})$ **and** $\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} | v_i,\, \mathbf{x}_i;\, v_p,\, \mathbf{x}_p\rangle(\mathbf{x})$**,** $(\mathbf{x} \in v_i \neq v_p)$ in the right-hand-side integral of Eq. (5.4), can be expressed in terms of stress-free strains $\boldsymbol{\beta}(\mathbf{x})$. Then, Eq. (5.4) can be rewritten in the index form, in view of Eqs. (3.7), (3.26), (4.5) and (4.14)

$$\langle \sigma_k \sigma_l \sigma_m \rangle_i(\mathbf{x}) = \langle \sigma_k \rangle_i(\mathbf{x}) \langle \sigma_l \rangle_i(\mathbf{x}) \langle \sigma_m \rangle_i(\mathbf{x}) + \Delta\sigma^2_{ikl}(\mathbf{x}) \langle \sigma_m \rangle_i(\mathbf{x}) + \Delta\sigma^2_{ikm}(\mathbf{x}) \langle \sigma_l \rangle_i(\mathbf{x}) + \Delta\boldsymbol{\sigma}^2_{ilm}(\mathbf{x}) \langle \sigma_k \rangle_i(\mathbf{x})$$
$$+ \Delta\sigma^3_{iklm}(\mathbf{x}), \tag{5.5}$$

where the Voigt notations have been used (k, l, m = 1,..., 6; i = 1, 2,...), and the tensor $\Delta\boldsymbol{\sigma}^3_i(\mathbf{x})$ is determined by a triple correlation function of stresses

$$\Delta\boldsymbol{\sigma}^3_i(\mathbf{x}) \equiv \int \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\, \boldsymbol{\beta}_1)\bar{v}_p \otimes \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\, \boldsymbol{\beta}_1)\bar{v}_p \otimes \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\, \boldsymbol{\beta}_1)\bar{v}_p \varphi(v_p,\, \mathbf{x}_p|;\, v_i,\, \mathbf{x}_i)\mathrm{d}\mathbf{x}_p$$
$$+ \int \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\, \boldsymbol{\beta}_1)\bar{v}_p \varphi(v_p,\, \mathbf{x}_p;\, v_i,\, \mathbf{x}_i) \otimes \Bigg\{ \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p)\boldsymbol{\beta}^{(p)_1}\bar{v}_p \otimes \int \mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\, \boldsymbol{\beta}_1)\bar{v}_q$$
$$\times \big[\varphi(v_q,\, \mathbf{x}_q|;\, v_p,\, \mathbf{x}_p;\, v_i,\, \mathbf{x}_i) - \varphi(v_q,\, \mathbf{x}_q|;\, v_i,\, \mathbf{x}_i)\big]\mathrm{d}\mathbf{x}_q + \int \mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\, \boldsymbol{\beta}_1)\bar{v}_q\big[\varphi(v_q,\, \mathbf{x}_q|;\, v_p,\, \mathbf{x}_p;\, v_i,\, \mathbf{x}_i)$$
$$- \varphi(v_q,\, \mathbf{x}_q|;\, v_i,\, \mathbf{x}_i)\big]\mathrm{d}\mathbf{x}_q \otimes \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\, \boldsymbol{\beta}_1)\bar{v}_p - \int \mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\, \boldsymbol{\beta}_1)\bar{v}_q \otimes \mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\, \boldsymbol{\beta}_1)\bar{v}_q \tag{5.6}$$
$$\times \varphi(v_q,\, \mathbf{x}_q|;\, v_i,\, \mathbf{x}_i)\mathrm{d}\mathbf{x}_q - \int\int \mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\, \boldsymbol{\beta}_1)\bar{v}_q \otimes \mathbf{T}_r(\mathbf{x}-\mathbf{x}_r,\, \boldsymbol{\beta}_1)\bar{v}_r \varphi(v_q,\, \mathbf{x}_q|;\, v_i,\, \mathbf{x}_i)$$
$$\times \big[\varphi(v_q,\, \mathbf{x}_q|;\, v_r,\, \mathbf{x}_r, v_i,\, \mathbf{x}_i) - \varphi(v_q,\, \mathbf{x}_q|;\, v_i,\, \mathbf{x}_i)\big]\mathrm{d}\mathbf{x}_q\, \mathrm{d}\mathbf{x}_r + \int \mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\, \boldsymbol{\beta}_1)\bar{v}_q[\varphi(v_q,\, \mathbf{x}_q|;\, v_p,\, \mathbf{x}_p;\, v_i,\, \mathbf{x}_i)$$
$$- \varphi(v_q,\, \mathbf{x}_q|;\, v_i,\, \mathbf{x}_i)]\mathrm{d}\mathbf{x}_q \otimes \int \mathbf{T}_q(\mathbf{x}-\mathbf{x}_q,\, \boldsymbol{\beta}_1)\bar{v}_q\big[\varphi(v_q,\, \mathbf{x}_q|;\, v_p,\, \mathbf{x}_p;\, v_i,\, \mathbf{x}_i) - \varphi(v_q,\, \mathbf{x}_q|;\, v_i,\, \mathbf{x}_i)\big]\mathrm{d}\mathbf{x}_q \Bigg\}\mathrm{d}\mathbf{x}_p.$$

In such a manner, the problem is reduced to calculating the multiple integrals on the right-hand-side of Eq. (5.6). These integrals have the different orders with respect to c under the dilute concentration of inclusions c. Taking into account only the principal part of the expansion Eq. (5.6), which is of the first order over c, we can write

$$\Delta\boldsymbol{\sigma}^3_i(\mathbf{x}) = \int \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\, \boldsymbol{\beta}_1)\bar{v}_p \otimes \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\, \boldsymbol{\beta}_1)\bar{v}_p \otimes \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\, \boldsymbol{\beta}_1)\bar{v}_p \varphi(v_p,\, \mathbf{x}_p|;\, v_i,\, \mathbf{x}_i)\mathrm{d}\mathbf{x}_p. \tag{5.7}$$

The omitted terms in Eq. (5.6) are of the second and third order in c.

It is interesting to consider a distribution law of real stresses different from the Gaussian distribution inside the inclusions. For this purpose some familiar relations are represented in Appendix A. One denotes the difference of the moments of the real random variable, $\langle \mathbf{e} \otimes \ldots \otimes \mathbf{e} \rangle$ and its Gaussian approximation, $\langle \mathbf{e}^{\mathrm{G}} \otimes \ldots \otimes \mathbf{e}^{\mathrm{G}} \rangle$ Eq. (A1) by

$$\Delta^{\mathrm{G}} \langle \mathbf{e} \otimes \ldots \otimes \mathbf{e} \rangle \equiv \langle \mathbf{e} \otimes \ldots \otimes \mathbf{e} \rangle - \langle \mathbf{e}^{\mathrm{G}} \otimes \ldots \otimes \mathbf{e}^{\mathrm{G}} \rangle, \tag{5.8}$$

where the Gaussian approximation $\mathbf{e}^G$ has a probability density (Eq. (A1); see Appendix A) obtained by the use of the first and second statistical moments of the random variable **e**. Then the comparison of Eqs. (A2) and (5.5) shows that

$$\Delta^G \langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle_i(\mathbf{x}) = \Delta \boldsymbol{\sigma}_i^3(\mathbf{x}). \tag{5.9}$$

It can be shown that a correction of n-order to $\langle \boldsymbol{\sigma}^G(\otimes\boldsymbol{\sigma}^G)^{n-1}\rangle_i(\mathbf{x})$ has the following linear principal part with respect to c under dilute c ($\mathbf{x} \in v_i$; $i = 0, 1,\ldots$; $n = 3, 4,\ldots$)

$$\Delta \boldsymbol{\sigma}^{n_i}(\mathbf{x}) \equiv \langle \boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{n-1}\rangle_i(\mathbf{x}) - \langle \boldsymbol{\sigma}^G(\otimes\boldsymbol{\sigma}^G)^{n-1}\rangle_i(\mathbf{x}) = \int \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p\Big[\otimes \mathbf{T}_p(\mathbf{x}-\mathbf{x}_p, \boldsymbol{\beta}_1) \times \bar{v}_p\Big]^{n-1}\varphi(v_p, \mathbf{x}_p|; v_i, \mathbf{x}_i)\mathrm{d}\mathbf{x}_p. \tag{5.10}$$

It should be noted that for the two component composites with isotropic homogeneous inclusions, $\boldsymbol{\beta}_1(\mathbf{x}) \equiv \beta_{10}(x)\boldsymbol{\delta}$, $\boldsymbol{\delta} \equiv \delta_{kl}$ and for the step binary probability density,

$$\varphi(v_p, \mathbf{x}_p|; v_i, \mathbf{x}_i) = \Big[1 - V^0_{ip}(\mathbf{x}_p)\Big]n^{(p)} \tag{5.11}$$

the normalized values of $\Delta\boldsymbol{\sigma}_i^n(\mathbf{x})$ for $n = 2$ (Eqs. (4.7) and (4.8)) and $n = 3, 4,\ldots$:

$$\Delta \boldsymbol{\sigma}_i^{n\ \mathrm{NORM}}(\mathbf{x}) \equiv \big[\Delta\boldsymbol{\sigma}_i^n(\mathbf{x})/\big(c\beta_{10}^n\big)\big]^{1/n} \tag{5.12}$$

are invariants with respect to the transformation fields $\boldsymbol{\beta}(\mathbf{x})$ and a given inclusion concentration.

5.2. Statistical moments of stresses inside the matrix in the vicinity of inclusions

We obtained the expressions of stress moments of arbitrary degree averaging over the volume of the matrix. But local characteristics of stress field are of prime interest in applications. Let us analyze the distribution of stresses in the matrix in the vicinity of an inclusion boundary with (see e.g Buryachenko and Kreher, 1995)

$$\boldsymbol{\sigma}_i^-(\mathbf{n}) = \boldsymbol{\sigma}_i^+(\mathbf{x}) - \boldsymbol{\Gamma}(\mathbf{n})\boldsymbol{\beta}_1^{(i)}, \tag{5.13}$$

where $\boldsymbol{\sigma}_i^-(\mathbf{n})$ and $\boldsymbol{\sigma}_i^+(\mathbf{x})$ are the random limiting stress outside and inside, respectively, near the inclusion boundary ∂v_i: $\boldsymbol{\sigma}_i^-(\mathbf{n}) = \lim \boldsymbol{\sigma}(\mathbf{y})$, $\boldsymbol{\sigma}^+(\mathbf{x}) = \lim \boldsymbol{\sigma}(\mathbf{z})$, $\mathbf{y} \to \mathbf{x}$, $\mathbf{z} \to \mathbf{x}$, $\mathbf{y} \in v^{(0)}$, $\mathbf{z} \in v_i$, $\mathbf{x} \in \partial v_i$; $\mathbf{n}$ is the unit outward normal vector on ∂v_i; the tensor $\boldsymbol{\Gamma}(\mathbf{n})$ is represented, e.g in Buryachenko and Kreher (1995). From Eq. (5.13), we notice that the average stress over ensemble realization in the matrix in the vicinity of the inclusion are generated from $\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})$ by parallel translation to the vector $\boldsymbol{\Gamma}(\mathbf{n})\boldsymbol{\beta}_1^{(i)}$, which does not depend on the form and concentration of inclusions. Then we obtain a relation for any moments of stresses $\boldsymbol{\sigma}^-(\mathbf{n})$ ($m = 1, 2,\ldots$)

$$\langle \boldsymbol{\sigma}^-(\mathbf{n})[\otimes\boldsymbol{\sigma}^-(\mathbf{n})]^{m-1}\rangle_x = \sum_{k=0}^{m} C_m^k \langle \boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{k-1}\rangle_i(\mathbf{x})\big\{\otimes\big[-\boldsymbol{\Gamma}(\mathbf{n})\boldsymbol{\beta}^{(i)_1}(\mathbf{x})\big]\big\}^{m-k}, \tag{5.14}$$

where $C_m^k \equiv \binom{k}{m}$ are the binomial coefficients, and the values $\langle\boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{k-1}\rangle_i(\mathbf{x})$ ($k = 1,2,\ldots, m$) are defined by Eqs. (4.5) and (5.10); in Eq. (5.14), it is necessary to take into account that $\langle\boldsymbol{\sigma}(\otimes\boldsymbol{\sigma})^{-1}\rangle_i(\mathbf{x}) \equiv 1$.

6. Correlation function of stresses

A conceptual sketch of a calculation of n-point moment of the stress field

$$\langle \boldsymbol{\sigma}(\mathbf{x}_1) \otimes \ldots \otimes \boldsymbol{\sigma}(\mathbf{x}_n)|v_1, \mathbf{x}_1; \ldots; v_n, \mathbf{x}_n; v_{n+1}, \mathbf{x}_{n+1}; \ldots; v_m, \mathbf{x}_m\rangle,$$

has been proposed by Buryachenko (1987a, 1987b) by the use of MEFM. This problem can be simplified for homogeneous elastic materials with heterogeneous stress free strains, because, in this case, the effective field, $\bar{\boldsymbol{\sigma}}(\mathbf{x})$, $\mathbf{x} \in v$ Eq. (3.4), does not depend on the transformation field, $\boldsymbol{\beta}_1^{(i)}(\mathbf{x})$, inside the inclusion. Now we turn our attention to the estimation of 2-point second moment of stresses $\langle \boldsymbol{\sigma}(\mathbf{x}) \otimes \boldsymbol{\sigma}(\mathbf{y})|v_i, \mathbf{x}_i; v_j, \mathbf{y}_j\rangle$ $(\mathbf{x} \in v_i,\ \mathbf{y} \in v_j,\ v_i \neq v_j)$. For this purpose, we take the tensor product of values of the stress Eq. (3.3) at different points $\mathbf{x} \in v_i$ and $\mathbf{y} \in v_j$

$$\boldsymbol{\sigma}(\mathbf{x}) \otimes \boldsymbol{\sigma}(\mathbf{y}) = \Big\{ \boldsymbol{\sigma}^0 - \mathbf{Q}_i(\mathbf{x}, \boldsymbol{\beta}_1) + \mathbf{T}_j(\mathbf{x} - \mathbf{x}_j, \boldsymbol{\beta}_1)\bar{v}_j + \int \Gamma(\mathbf{x} - \mathbf{x}_p)[\boldsymbol{\beta}_1^{(p)} V(\mathbf{x}_p|; v_i, \mathbf{x}_i; v_j, \mathbf{x}_j) - \langle \boldsymbol{\beta}_1 \rangle] \mathrm{d}\mathbf{x}_p \Big\} \otimes \boldsymbol{\sigma}(\mathbf{y}). \tag{6.1}$$

Then, if one averages Eq. (6.1) over the ensemble realization under the fixed inclusions v_i and v_j, one obtains $(\mathbf{x} \in v_i \neq v_j,\ \mathbf{y} \in v_j)$

$$\begin{aligned}\langle \boldsymbol{\sigma}(\mathbf{x}) \otimes \boldsymbol{\sigma}(\mathbf{y})|v_i, \mathbf{x}_i; v_j, \mathbf{x}_j\rangle &= \Big[- \mathbf{Q}_i \boldsymbol{\beta}_1^{(i)} + \mathbf{T}_j(\mathbf{x} - \mathbf{x}_j, \boldsymbol{\beta}_1)\bar{v}_j \Big] \otimes \langle \boldsymbol{\sigma}|v_j, \mathbf{x}_j; v_i, \mathbf{x}_i\rangle(\mathbf{y}) \\ &+ \int [\mathbf{T}_p(\mathbf{x} - \mathbf{x}_p)\boldsymbol{\beta}_1^{(p)} v_p \langle \boldsymbol{\sigma}|v_j, \mathbf{x}_j; v_i, \mathbf{x}_i; v_p, \mathbf{x}_p\rangle(\mathbf{y}) \varphi(v_p, \mathbf{x}_p|; v_i, \mathbf{x}_i; v_j, \mathbf{x}_j) \\ &- \Gamma(\mathbf{x} - \mathbf{x}_p)\langle \boldsymbol{\beta}_1 \rangle \langle \boldsymbol{\sigma}|v_j, \mathbf{x}_j; v_i, \mathbf{x}_i\rangle(\mathbf{y})] \mathrm{d}\mathbf{x}_p.\end{aligned} \tag{6.2}$$

We transform (6.2) by the use of Eqs. (3.26) and (3.28), in which $n = 3$ gives

$$\begin{aligned}\langle \boldsymbol{\sigma}(\mathbf{x}) \otimes \boldsymbol{\sigma}(\mathbf{y})|v_i, \mathbf{x}_i; v_j, \mathbf{x}_j\rangle &= \langle \boldsymbol{\sigma}|v_i, \mathbf{x}_i; v_j, \mathbf{x}_j\rangle(\mathbf{x}) \otimes \langle \boldsymbol{\sigma}|v_j, \mathbf{x}_j; v_i, \mathbf{x}_i\rangle(\mathbf{y}) + \int \mathbf{T}_p(\mathbf{x} - \mathbf{x}_p, \boldsymbol{\beta}_1) \\ &\times \bar{v}_p \otimes \mathbf{T}_p(\mathbf{y} - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p] \varphi(v_p, \mathbf{x}_p|; v_i, \mathbf{x}_i; v_j, \mathbf{x}_j) \mathrm{d}\mathbf{x} + \int\int [\mathbf{T}_p(\mathbf{x} - \mathbf{x}_p, \boldsymbol{\beta}_1)\bar{v}_p] \otimes [\mathbf{T}_q(\mathbf{y} - \mathbf{x}_q, \boldsymbol{\beta}_1) \\ &\times \bar{v}_q] \varphi(v_p, \mathbf{x}_p|; v_i, \mathbf{x}_i; v_j, \mathbf{x}_j) + \big[\varphi(v_q, \mathbf{x}_q|; v_p, \mathbf{x}_p; v_i, \mathbf{x}_i; v_j, \mathbf{x}_j) - \varphi(v_q, \mathbf{x}_q|; v_i, \mathbf{x}_i; v_j, \mathbf{x}_j)\big] \mathrm{d}\mathbf{x}_q \, \mathrm{d}\mathbf{x}_p.\end{aligned} \tag{6.3}$$

Now let us obtain the relation for the correlation function by the use of an alternative method proposed by Buryachenko and Kreher (1995) for an analysis of elastically inhomogeneous media. Then, at $\mathbf{x} \in v_i$, and $\mathbf{x}_i \neq \boldsymbol{x}_{\mathrm{j}}$,

$$\boldsymbol{\sigma}(\mathbf{x}|v_i, \mathbf{x}; v_j, \mathbf{x}_j) = \tilde{\boldsymbol{\sigma}}_{ij}(\mathbf{x}) - \mathbf{Q}_i(\mathbf{x}, \boldsymbol{\beta}_1) + \mathbf{T}_j(\mathbf{x} - \mathbf{x}_j, \boldsymbol{\beta}_1)\bar{v}_j, \tag{6.4}$$

where $\tilde{\boldsymbol{\sigma}}_{ij}$ is a random effective field, in which two considered inclusions, v_i and v_j, are located. It should be mentioned that the right-hand-side of Eq. (6.4) does not depend on the value of the effective field, $\tilde{\boldsymbol{\sigma}}_{ij}(\mathbf{y})$ $(\mathbf{y} \in v_j)$, inside inclusion v_j. Thanks to this fact, the closing assumption Eq. (7.5) from Buryachenko and Kreher (1995) degenerates into $(\mathbf{x} \in v_i,\ \mathbf{y} \in v_j)$

$$\langle \big[\tilde{\boldsymbol{\sigma}}_{ij}(\mathbf{x}) - \mathbf{Q}_i(\mathbf{x}, \boldsymbol{\beta}_1)\big] \otimes [\tilde{\boldsymbol{\sigma}}_{ij}(\mathbf{y}) - \mathbf{Q}_j(\mathbf{y}, \boldsymbol{\beta}_1)] \rangle = \langle \boldsymbol{\sigma} \rangle_i(\mathbf{x}) \otimes \langle \boldsymbol{\sigma} \rangle_j(\mathbf{y}). \tag{6.5}$$

Then, the 2-point second moment of stress inside the inclusions, v_i and v_j, is obtained by taking the tensor product of $\boldsymbol{\sigma}(\mathbf{x}|\mathbf{x}_i;\mathbf{x}_j)$ Eq. (6.4) into a related variable, $\boldsymbol{\sigma}(\mathbf{y}|\mathbf{x}_j;\mathbf{x}_i)$ Eq. (6.4) in view of closure Eq. (6.5)

$$\langle\boldsymbol{\sigma}(\mathbf{x})\otimes\boldsymbol{\sigma}(\mathbf{y})\rangle = [\langle\boldsymbol{\sigma}\rangle_i(\mathbf{x}) + \mathbf{T}_j(\mathbf{x}-\mathbf{x}_j,\ \boldsymbol{\beta}_1)v_j]\otimes[\langle\boldsymbol{\sigma}\rangle_j(\mathbf{y}) + \mathbf{T}_i(\mathbf{y}-\mathbf{x}_i,\ \boldsymbol{\beta}_1)v_i]. \tag{6.6}$$

This expression should be confronted with the previously obtained relation Eq. (6.3). Early in the game, it should be recorded that the correlation function can result from the simple formula of the first order approximation

$$\langle\boldsymbol{\sigma}(\mathbf{x})\otimes\boldsymbol{\sigma}(\mathbf{y})\rangle = \langle\boldsymbol{\sigma}\rangle_i(\mathbf{x})\otimes\langle\boldsymbol{\sigma}\rangle_j(\mathbf{y}). \tag{6.7}$$

Eq. (6.7) does not take into account the binary interaction between inclusions $\mathbf{x}\in v_i$ and $\mathbf{y}\in v_j$. This interaction is considered by the second order approximation (6.6). The result of the third order approximation (6.3) is determined by the action of an arbitrary third inclusion v_p. The correction,

$$\Delta\langle\boldsymbol{\sigma}(\mathbf{x})\otimes\boldsymbol{\sigma}(\mathbf{y})\rangle \equiv \langle\boldsymbol{\sigma}(\mathbf{x})\otimes\boldsymbol{\sigma}(\mathbf{y})|v_i,\ \mathbf{x}_i;\ v_j,\ \mathbf{x}_j\rangle - \langle\boldsymbol{\sigma}|v_i,\ \mathbf{x}_i;\ v_j,\ \mathbf{x}_j\rangle(\mathbf{x})\otimes\langle\boldsymbol{\sigma}|v_j,\ \mathbf{x}_j;\ v_i,\ \mathbf{x}_i\rangle(\mathbf{y}), \tag{6.8}$$

can be written, in view of only the principal part of Eq. (6.3), as

$$\Delta\langle\boldsymbol{\sigma}(\mathbf{x})\otimes\boldsymbol{\sigma}(\mathbf{y})\rangle = \int[\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p,\ \boldsymbol{\beta}_1)\bar{v}_p]\otimes\mathbf{T}_p(\mathbf{y}-\mathbf{x}_p,\ \boldsymbol{\beta}_1)\bar{v}_p\varphi(v_p,\ \mathbf{x}_p|;\ v_i,\ \mathbf{x}_i;\ v_j,\ \mathbf{x}_j)\mathrm{d}\mathbf{x}_p, \tag{6.9}$$

under the considered fixed inclusions, v_i and v_j.

7. Numerical results

As an example, we consider a Si_3N_4 composite with isotropic components $\mathbf{L} = (3k,\ 2\mu) \equiv 3k\mathbf{N}_1 + 2\mu\mathbf{N}_2$, $\mathbf{N}_1 \equiv \boldsymbol{\delta}\otimes\boldsymbol{\delta}/3$, $\mathbf{N}_2 \equiv \mathbf{I}-\mathbf{N}_1$ containing the identical SiC spherical inclusion. We will use the following elastic constants and thermal expansion coefficients, as usually found in the literature (see, e.g. Kreher and Janssen, 1992), and displayed in Table 1.

The thermal strains have been calculated with the assumption that internal stresses do not relax by creep of components below T_0. For an assessment of stress, a temperature difference $(T-T_0)$ of about 1000 K between room temperature and the stress-free state at the elevated temperature is assumed ($\boldsymbol{\beta} \equiv \mathbf{a}^T(T-T_0)$, $\mathbf{a}^T = a^T\delta_{ij}$).

For a low volume fraction of SiC, one may describe the composite as a Si_3N_4 matrix with SiC particles embedded in it. The SiC particles are assumed to be spheres with radius a. Two alternative radial functions of inclusion distribution will be examined:

$$g(\mathbf{x}_i-\mathbf{x}_j) \equiv \varphi(v_i,\ \mathbf{x}_i|;\ v_j,\ \mathbf{x}_j)/n_i = H(r-2a),\ r \equiv |\mathbf{x}_i-\mathbf{x}| \tag{7.1}$$

Table 1
Thermoelastic constants

	k (GPa)	μ (GPa)	a^T (10^{-6} K)
Si_3N_4	236.4	121.9	3.4
SiC	208.3	169.5	4.4

and (see Willis, 1978)

$$g(\mathbf{x}_i - \mathbf{x}_j) = H(r - 2a)\left\{1 + \left[\frac{2 + c}{2(1 - c)^2} - 1\right]\cos\left(\frac{\pi r}{a}\right)e^{2(2-r/a)}\right\}, \tag{7.2}$$

where H denotes the Heaviside step function, $r \equiv |\mathbf{x}_i - \mathbf{x}_j|$ is a distance between the nonintersecting inclusions v_i and v_j and c is the volume fraction of SiC.

For the representation of numerical results in dimensionless form, we define the normalizing coefficient $\tau \equiv -3Q^k \boldsymbol{\beta}_{10}$, where $3Q^k$ equals the bulk component of the tensor $\mathbf{Q} = (3Q^k, 2Q^\mu)$. The physical meaning of τ follows from Eq. (3.17), according to which, τ equals the component of hydrostatic stress inside a single isolated inclusion in an infinite homogeneous matrix (at $\mathbf{M}(\mathbf{x}) \equiv$ const.); for our specific composite, SiC–Si_3N_4, we have $\tau = 289$ MPa and Poisson's ratio $\nu = 0.28$. In order to carry out the needed numerical estimates, we will use the expressions of the tensors, $\mathbf{Q}_p$, $\mathbf{T}_p(\mathbf{x}-\mathbf{x}_p)$, $\boldsymbol{\Gamma}(\mathbf{n})$ Eq. (5.13) (p = 1, 2,...), which are presented, e.g. in Buryachenko and Kreher (1995).

At first, we will estimate the effect of the assumption of elastic homogeneity of the materials ($\mathbf{M}(\mathbf{x}) \equiv \mathbf{M}^{(0)}$). The results $\langle \boldsymbol{\sigma}_{11} \rangle_1 \sim c$ outlined in Fig. 1 are calculated by both the exact relation Eq. (3.8) ($\mathbf{M}(\mathbf{x}) \equiv \mathbf{M}^{(0)}$) and by the approximate MEFM by Buryachenko and Kreher (1995), with ($\mathbf{M}(\mathbf{x}) \neq$ const). Fig. 2 shows the mean square deviation of stresses inside the inclusion, $|\Delta \boldsymbol{\sigma}^2_{1|1111}|^{0.5}$, as a function of the standard deviation of inclusion concentration $(\Delta c^2)^{0.5} \equiv [c(1-c)]^{0.5}$, which were calculated by the perturbation method Eq. (3.22) under the assumption of nonhomogeneity ($\mathbf{M}(\mathbf{x}) \neq$ const.) and homogeneity ($\mathbf{M}(\mathbf{x}) \equiv \mathbf{M}^{(0)}$) of the elastic properties of the composite. From Figs. 1 and 2, we notice that the error of the assumption of homogeneity of elastic properties of the ceramic is about the error caused by ignoring the radial distribution function Eq. (7.2). Because of this, below we will consider only the case $\mathbf{M}(\mathbf{x}) \equiv \mathbf{M}^{(0)}$.

Let us evaluate a perturbation $\delta\boldsymbol{\sigma}_1(\mathbf{x}) \equiv \langle \boldsymbol{\sigma} | v_1, \mathbf{x}_1; v_2, \mathbf{x}_2 \rangle(\mathbf{x}) - \langle \boldsymbol{\sigma} \rangle_1(\mathbf{x})$, $\mathbf{x} = \mathbf{x}_1$, caused by the inclusion v_2 at the center of the inclusion v_1; $\mathbf{x}_1 = (0, 0, 0)$, $\mathbf{x}_2 = (r, 0, 0)$. We will consider triple conditional probability densities

$$\varphi(v_p, \mathbf{x}_p |; v_1, \mathbf{x}_1; v_2, \mathbf{x}_2) = n_1 g(\mathbf{x}_p - \mathbf{x}_1) g(\mathbf{x}_p - \mathbf{x}_2) H(|\mathbf{x}_p - \mathbf{x}_1| - 2a) H(|\mathbf{x}_p - \mathbf{x}_2| - 2a) \tag{7.3}$$

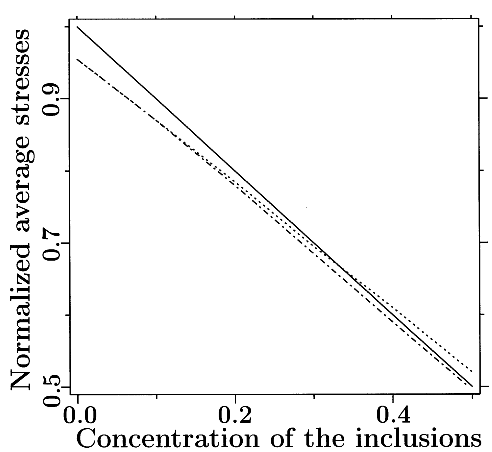

Fig. 1. Normalized average stresses within the inclusions $\langle \sigma_{11} \rangle_1 / \tau$ calculated by the exact relation Eq. (3.17) under $\mathbf{M}(\mathbf{x}) \equiv \mathbf{M}^{(0)}$ (solid line) and by the MEFM (under $\mathbf{M}(\mathbf{x}) \neq$ const.) for the step correlation function Eq. (7.1) and the real one Eq. (7.2) (dotted and dot–dashed lines, respectively).

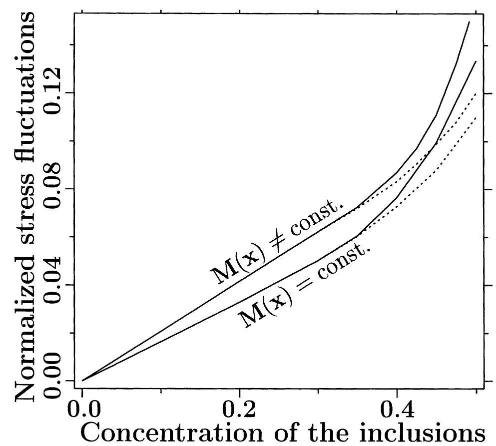

Fig. 2. Normalized stress fluctuation $|\Delta\sigma^2_{1|1111}|^{0.5}/\tau$ as a function of $(\Delta c^2)^{0.5}$ calculated for step correlation function Eq. (7.1) (dotted lines) and real one Eq. (7.2) (solid lines) under the assumption of nonhomogeneity ($\mathbf{M}(\mathbf{x}) \neq$ const.) and homogeneity ($\mathbf{M}(\mathbf{x})=$ const.) of elastic properties.

and binary ones (Eqs. (7.1) and (7.2)). The impossibility of the inclusion intersection is taken into account by the Heaviside step function H. A neighboring order in the triple point distribution Eq. (3.26) is constructed by the use of the binary probability density; a selection of more complicated distribution functions will not be considered. Fig. 3 shows the curves which are calculated for $c = 0$, 0.1 and 0.4 by the use of Eqs. (3.26) and (7.3) for the radial distribution functions, Eqs. (7.1) and (7.2). It is evident from these curves that considering only the principal part in the action of the second inclusion, $\mathbf{T}_2(\mathbf{x}_1 - \mathbf{x}_2)\boldsymbol{\beta}_1^{(2)}\bar{v}_2$, leads to significant errors under nondilute inclusion concentration. The non-monotonical character of curves 3 and 4 near the point $|\mathbf{x}_1 - \mathbf{x}_2| = 4a$ is explained by the occurance of the nonzero probability of the location of some inclusion v_p between fixed inclusions v_1 and v_2, under $r/a > 4$ only.

We come now to the estimation of the nonhomogeneity of stress fluctuations $\Delta\boldsymbol{\sigma}_1^2(\mathbf{x})$ inside the

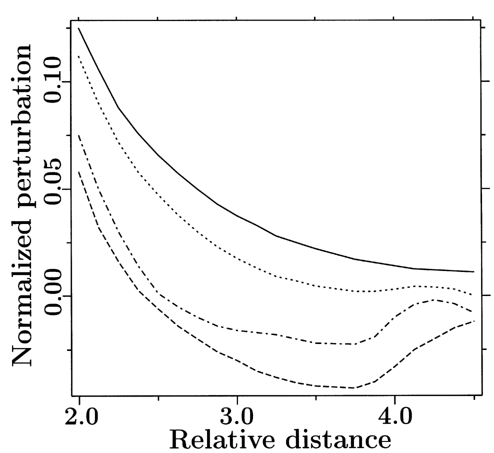

Fig. 3. Normalized perturbation $\delta\sigma_{1|11}(\mathbf{x})$ produced by the inclusion v_2 into a center of the inclusion v_1 for $c = 0$ (solid line), 0.1 (dotted line) and 0.4 (dot–dashed and dashed curves) as a function of the relative distance $|\mathbf{x}_1 - \mathbf{x}_2|/a$. Dashed line is calculated for the step correlation function Eq. (7.1), and dot–dashed line is obtained for real correlation function Eq. (7.2).

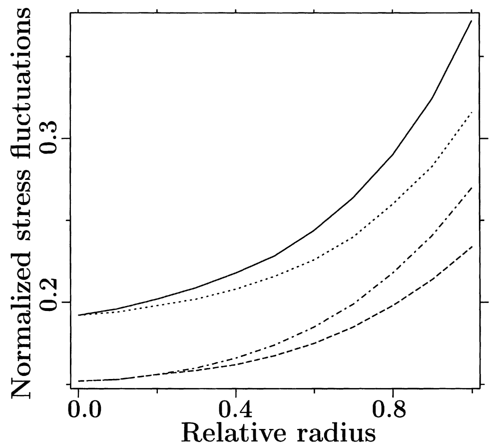

Fig. 4. Normalized stress fluctuations $|\Delta\sigma^2_{1|1111}(\mathbf{x})|^{0.5}/\tau$ (solid and dot–dashed lines) and $|\Delta\sigma_{1|3333}(\mathbf{x})|^{0.5}/\tau$ (dot and dashed lines) both for the real correlation function Eq. (7.2) (solid and dotted lines) and the step function Eq. (7.1) (dot–dashed and dashed lines) as a function of the relative radius r/a.

inclusions. The curves in Fig. 4 were calculated by the use of Eqs. (4.7) and (4.8) with $c = 0.4$ for radial distribution functions, Eqs. (7.2) and (7.1), respectively; $\mathbf{x} = (r, 0, 0)$. The normalized components $\Delta\sigma^2_{1|1111}(\mathbf{x})$ and $\Delta\sigma^2_{1|3333}(\mathbf{x})$ are displayed in Fig. 4 by solid and dashed curves, respectively. For the step correlation function Eq. (7.1), the calculated curves are invariant with respect to the inclusion concentration.

Let us estimate an influence of the binary fractional inclusion composition on the inhomogeneity of stress moments inside the inclusions; $c_1 = c_2 = 0.2$ and $a_1 = 2a$. Fig. 5 shows the normalized curves $|\Delta\boldsymbol{\sigma}^2_{1|1111}(\mathbf{x})|^{0.5}/\tau$, which are calculated by Eqs. (4.7), (4.8) and (7.1) inside the smaller inclusions and the larger one; $\mathbf{x} = (r, 0, 0)$. It is evident from the figure that stress fluctuations at the inclusion surfaces for different inclusion sizes agree very closely, but they differ from one another by a factor of 1.5 in the

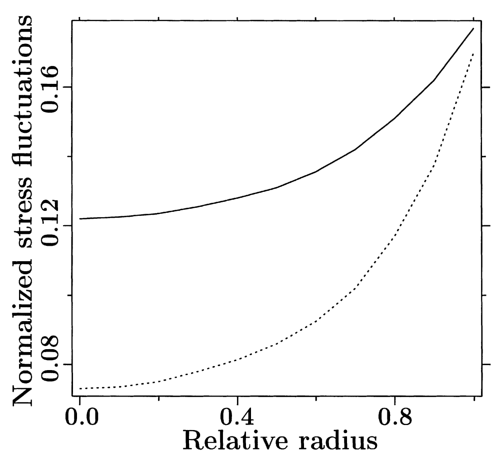

Fig. 5. Normalized stress fluctuations $|\Delta\sigma^2_{1|1111}(\mathbf{x})|^{0.5}/\tau$ inside both the small (solid line) and large inclusion (dotted line), as a function of the relative radius r/a; $\mathbf{x} = (r, 0, 0)$.

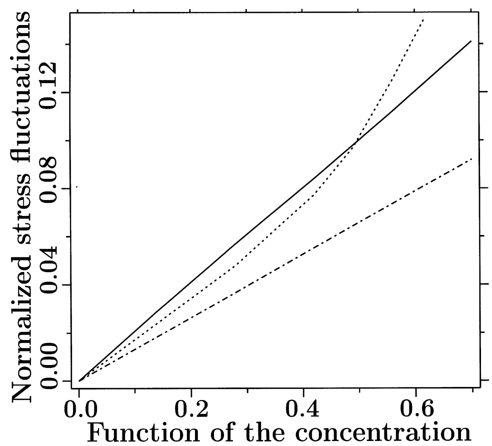

Fig. 6. Normalized stress fluctuations inside the inclusions $|\langle \Delta\sigma_{1111}\rangle_1|^{0.5}/\tau$ calculated by the use of the perturbation method Eq. (3.22) (dotted line) as well as by the integral equation method (Eqs. (4.7), (4.8) and (4.9)) with consideration of the H_1 hypothesis (dot–dashed line) and without one (solid line).

center of the inclusions, notwithstanding the fact that the average stresses inside the inclusions are invariant with respect to their size.

We compare the values of average fluctuations inside the inclusions $\langle \Delta\boldsymbol{\sigma}^2\rangle_1$, which are obtained by the different methods using the step distribution function Eq. (7.1). The results for stress fluctuations $|\langle \Delta\boldsymbol{\sigma}^2\rangle_1|^{1/2}/\tau \sim c^{1/2}$ shown in Fig. 6 are estimated by the use of the perturbation method Eq. (3.22) as well as of the method of integral equations (Eqs. (4.7), (4.8) and (4.9)) and with due regard for only the principal part (analogous to Eq. (4.7)) of the expansion Eq. (4.11) of the EFM. We notice that the abandonment of hypothesis H_1 in MEFM leads to important refinements of the numerical results. In so doing, the average stresses inside the inclusions $\langle\boldsymbol{\sigma}\rangle_i$, calculated by three different methods have the same values.

Let us estimate the normalized correction, $\Delta\boldsymbol{\sigma}_1^{n\ \mathrm{NORM}}(\mathbf{x}) \sim r$ Eq. (5.12), of different orders to the

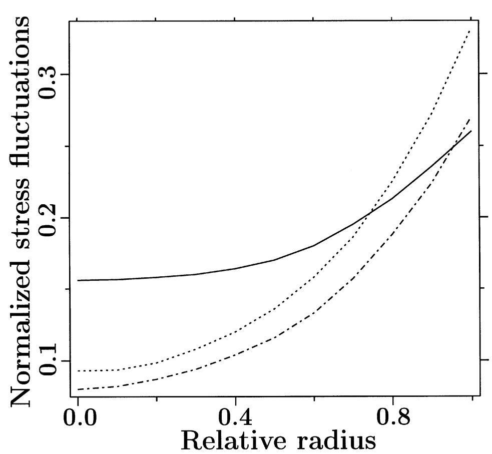

Fig. 7. The first components of n-order normalized correction $\Delta\boldsymbol{\sigma}_1^{n\ \mathrm{NORM}}(\mathbf{x})/3Q^k$ to the Gaussian approximation $\langle\boldsymbol{\sigma}^{\mathrm{G}}(\otimes\boldsymbol{\sigma}^{\mathrm{G}})^{n-1}\rangle(\mathbf{x})$: $n = 2$ (solid line), $n = 3$ (dot–dashed line), $n = 4$ (dotted line), as a function of the relative radius r/a; $\mathbf{x} = (r, 0, 0)$.

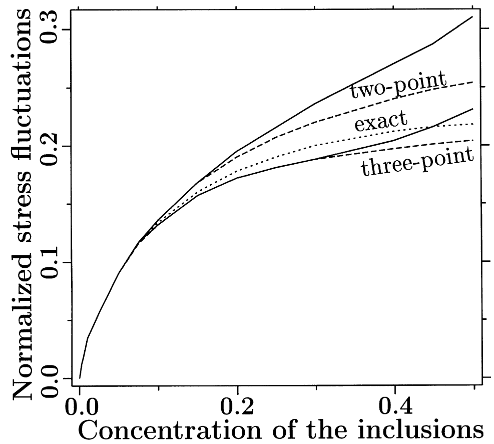

Fig. 8. Normalized stress fluctuations inside the components $|\Delta\bar{\sigma}^2_{1111}|^{0.5}/\tau$ calculated for both the step correlation function Eq. (7.1) (dashed lines) and the real one Eq. (7.2) (solid lines), which are obtained under the two-point and three-point approximations. The exact solution is plotted by the dotted line.

Gaussian approximation of the real stochastic stress distribution. The curves, $\Delta\boldsymbol{\sigma}_1^{n\ \mathrm{NORM}}(\mathbf{x}) \sim r$, in Fig. 7 were calculated under $n = 2, 3, 4$ and the radial distribution function Eq. (7.1) by the use of Eqs. (4.7) and (5.10) and the notation Eq. (5.12); we have considered only the components $\Delta\sigma^{n\ \mathrm{NORM}}_{1|i_1,\ldots}$, $i_n(\mathbf{x})$ $(i_1 = \ldots = i_n = 1)$ as the functions of the coordinate $\mathbf{x} \equiv (r, 0, 0)$.

Now we evaluate the accuracy of stress fluctuation in the components $\bar{\Delta}\boldsymbol{\sigma}^2$ Eq. (4.18) by use of both the the three-point approximation (4.16) and the two-point approximation (Eqs. (4.7) and (4.15)); it turns out that the contribution from the stress fluctuation inside the components, $\langle\Delta\sigma^2_{1111}\rangle_i$ $(i = 0, 1)$, is less than the exact relation for stress fluctuations in the whole composite, $\Delta\boldsymbol{\sigma}^2 \equiv \langle\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}\rangle - \langle\boldsymbol{\sigma}\rangle \otimes \langle\boldsymbol{\sigma}\rangle$ (defined by Ortiz and Molinari, 1988), by a factor 5. In Fig. 8, the curves of the two-point approximations were calculated by using the principal part of the expansion for $\langle\Delta\boldsymbol{\sigma}^2\rangle_1$ (Eqs. (4.7) and (4.9)) and $\langle\Delta\boldsymbol{\sigma}^2\rangle_0$ Eq. (4.15) (which are proportional to c under $c \to 0$) for the radial distribution functions (Eqs. (7.2) and (7.1)), respectively. A step function,

$$\varphi(v_p, \mathbf{x}_p|; v_0, \mathbf{x}_0) = H(|\mathbf{x} - \mathbf{x}_0| - a)n_1, \tag{7.4}$$

has been applied for calculation of these curves. For comparison, the exact result described by Eqs. (3.16) and (3.20), as well as two curves of three-point approximations calculated by Eq. (4.16), are also plotted in the same figure. One uses triple conditional probability densities,

$$\begin{aligned}&\varphi(v_q, \mathbf{x}_q|; v_p, \mathbf{x}_p; v_i, \mathbf{x}_i)\\ &\quad = n_q g(\mathbf{x}_q - \mathbf{x}_i) g(\mathbf{x}_p - \mathbf{x}_i) H(|\mathbf{x}_q - \mathbf{x}_i| - 2a) \cdot H(|\mathbf{x}_p - \mathbf{x}_i| - 2a) H(|\mathbf{x}_p - \mathbf{x}_q| - 2a)\end{aligned} \tag{7.5}$$

and

$$\varphi(v_q, \mathbf{x}_q|; v_p, \mathbf{x}_p; v_0, \mathbf{x}_0) = n_q H(|\mathbf{x}_q - \mathbf{x}_0| - a) H(|\mathbf{x}_p - \mathbf{x}_i| - a), \tag{7.6}$$

and the binary probability density (Eqs. (7.1), (7.2) and (7.4)). Curves of the three-point approximation are computed with the radial distribution functions (Eqs. (7.2) and (7.1)) as well. As may be seen from Fig. 8, the accuracy of the estimations may be substantially extended by taking into account a threefold interaction of the inclusions. The error of the calculation by the exact formula (4.16) is dictated by the

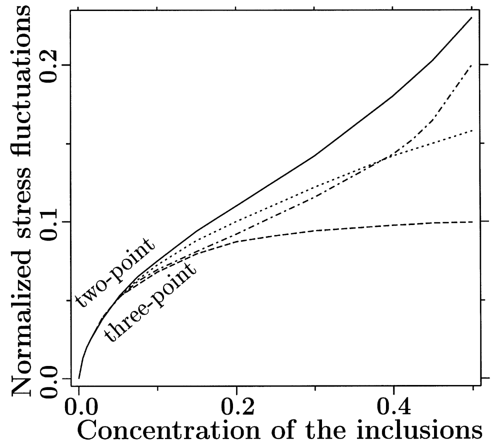

Fig. 9. Normalized stress fluctuations inside the inclusions $|\langle \Delta\sigma_{1111}\rangle_1|^{0.5}/\tau$ calculated by the use of two-point (solid and dotted lines) and three-point (dot–dashed and dashed lines) approximations for both the step correlation function Eq. (7.1) (dotted and dashed lines) and the real one Eq. (7.2) (solid and dashed lines).

inaccuracy of the determination of different conditional probability densities (Eqs. (7.1), (7.2), (7.3), (7.4), (7.5) and (7.6)). The curves presented are representative of averaging fluctuations of the stresses in the composite. The degree of dissimilarity between the different models should be expected from the calculation of stress fluctuations inside the inclusions. Really, the results $(\langle \Delta\sigma^{2_{1111}}\rangle_1)^{0.5} \sim c$, outlying Fig. 9, are calculated in an exact three-point relation Eq. (4.5) and an approximate two-point formula (4.7) when used with the correlation functions (Eqs. (7.3), (7.4), (7.5) and (7.6)), with the radial distribution functions, Eq. (7.2) (solid lines) and Eq. (7.1) (dashed lines). From this figure, we notice that the approximate formula (4.7) under the binary-step function Eq. (7.1) can be used for the estimation of stress fluctuations $\langle \Delta\boldsymbol{\sigma}^2\rangle_i$ with sufficient accuracy.

The results for $\Delta\langle\sigma_{11}(\mathbf{x})\otimes\sigma_{11}(\mathbf{y})\rangle$ and $\Delta\langle\sigma_{33}(\mathbf{x})\otimes\sigma_{33}(\mathbf{y})\rangle$ Eq. (6.9) are plotted in Fig. 10 after using the binary Eq. (7.1) and triple conditional probability densities Eq. (7.3) with $\mathbf{x}=(a, 0, 0)\in v_i$ and $\mathbf{y}=(r-a,$

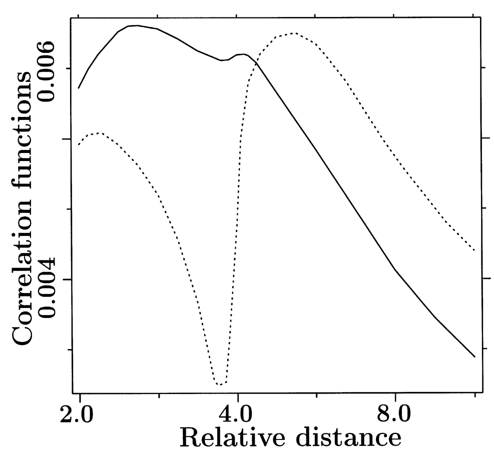

Fig. 10. Normed correlation functions $|\Delta\langle\sigma_{11}(\mathbf{x}_1)\otimes\sigma_{11}(\mathbf{x}_2)\rangle|^{0.5}/\tau$ (dashed line) and $|\Delta\langle\sigma_{33}(\mathbf{x}_1)\otimes\sigma_{33}(\mathbf{x}_2)\rangle|^{0.5}/\tau$ (solid line) as a function of the relative distance $|\mathbf{x}-\mathbf{y}|/a$ between the inclusions.

0, 0) $\in v_2$; the centers of fixed inclusions, v_1 and v_2, are located at the points $\mathbf{x}_1 = (0, 0, 0)$ and $\mathbf{x}_2 = (r, 0, 0)$. The non-monotonic character of the calculated curves are attributable to the fact that the probability of spacing of the third inclusion, v_p, between fixed inclusions, v_1 and v_2 under $r/a > 4$, does not vanish.

It should be mentioned that all normalized numerical results (Figs. 1–10) are obtained for $\nu = 0.28$, depend only on Poisson's ratio, and are invariant with respect to other thermoelastic properties of the components. But for practically important values, $0.2 < \nu < 0.35$, the calculated nonhomogeneous stress fluctuations along the radius of the inclusions (Figs. 4, 5 and 7) vary in magnitude by, at most, 20 per cent; analogous deviation for average stress fluctuation over each component equals 1 per cent. Therefore, the normalized results noted may be used with some caution for analysis of the sufficiently wide class of elastically homogeneous composites with the microtopology studied here.

8. Conclusion

Needless to say, the real purpose of the author, beyond the immediate scope of the considered problem, only seems to be theoretical. Stress fluctuations in the components of random structure composites represent a measure of inhomogeneity of stress fields in the components. The fundamental roles of such inhomogeneities described by the stress fluctuations are discussed in detail by Buryachenko (1996), Ponte Castañeda (1997), and Suquet (1997) for a wide class of nonlinear problems of micromechanics, such as nonlinear elasticity, viscosity and creeping, elastoplastisity, and strength. The principal advantages of the proposed method of integral representations for stress fluctuations (Eqs. (4.5), (4.15) and (5.13)), in comparison with the perturbation method and some others, were shown by Buryachenko and Rammerstorfer (1997) for the purely isothermal elastic case ($\boldsymbol{\beta} = \mathbf{0}$) in a composite with uncoated inclusions. For the thermoelastic case, additional advantages as well as other interesting aspects when employing the proposed method should be mentioned (see Buryachenko and Rammerstorfer, 1998a).

For elastoplastic analyses based on estimations of some nonlinear functions of local stresses, e.g. the yield condition, taking stress inhomeneities in the components into account, it is very popular to use secant and tangent moduli concepts (see the references in the indicated papers). This way, the nonlinear problem at each solution increment reduces to the averaging linear elastic problem with $\boldsymbol{\beta} \equiv \mathbf{0}$. The use of the secant modulus concept creates the known complications since, generally, the local stress state is not monotonical and proportional, even with monotonical and proportional external loading. The tangent moduli concept leads to the necessity of also considering the matrix material as being anisotropic at each solution step. This would not lead to any problem in the framework of the 'quasi-crystalline' approximation by Lax (1951) (see also Buryachenko and Rammerstorfer, 1997), but it leads to some computing difficulties at the realization of the MEFM for which advantages in comparison with some popular methods were justified (see, for references, Buryachenko, 1996; Buryachenko and Rammerstorfer, 1997). However, the integral representations for stress fluctuations (Eqs. 4.5 and 4.15) permit the use of the incremental method with fixed elastic properties of the components and with accumulating plastic strains ($\boldsymbol{\beta}^{(i)} \neq$ const.). This method was presented by Buryachenko and Rammerstorfer (1996a) (see also Buryachenko, 1999) for elastically homogeneous materials with a thermal mismatch of the components. The analysis of the elastic mismatch will be pursued in a forthcoming work of the author. Another improvement is connected with the abandonment of the assumption of homogeneity of plastic strains in the matrix. In so doing, the concentration of plastic strains in the vicinity of inclusions plays the role of a 'coating', exhibiting an inhomogeneous transformation field along the inclusion surfaces. This model was realized by Buryachenko et al. (1997) in the framework of the mean field method by Dvorak (1993), and can be generalized with regard to stress fluctuations in the components as found from Eqs. (4.5) and (4.15).

Moreover the method of integral equations proposed also has qualitative benefits following immediately from the consideration of multiparticle interactions. From such considerations, it can be concluded that the final relations for statistical moments of stresses (Eqs. (3.7), (4.5) and (4.15)) depend explicitly not only on the local concentration of the inclusions, but also on at least binary correlation functions of the inclusions. Therefore, for statistically inhomogeneous composites (the so-called 'graded' materials), the local statistical moments of stresses in the components are nonlocal functions of the inclusion concentration. In the framework of effective field hypothesis (see for details, e.g. Buryachenko and Kreher, 1995), this nonlocal effect was shown by Buryachenko and Rammerstorfer (1998b, 1999c) (the case, $\mathbf{M}$, $\boldsymbol{\beta} \neq \text{const}$) under estimation of average stresses in the inclusions arranged in a finite cloud in the infinite matrix. For the case $\mathbf{M} \equiv \text{const.}$, $\boldsymbol{\beta} \neq \text{const.}$, it was exactly determined that the conditional average inside the inclusions is changed by over 25 per cent in the boundary layer of the cloud. Similar nonlocal effects are expected under the estimation of second moment of stresses by the use of Eqs. (4.5) and (4.15). However, more detailed considerations of these effects are beyond the scope of current paper.

Finally, it should be particularly emphasized that the representations for statistical moments of stresses (Eqs. (3.7), (3.26), (4.5), (4.15) and (5.6)) obtained are new and exact. No restrictions are imposed on the shape and microtopology of inclusions (including coated structure) as well as on the inhomogeneity of the stress state in the inclusions. For some additional assumptions in the concrete examples, it was shown that the second moment of stresses inside inclusions are essentially inhomogeneous functions and changed by a factor 2 or more along the inclusion radius (see Fig. 4). Moreover, even under the evaluation of average statistical moments of stresses inside the constituents, the method proposed makes it possible to improve the estimation by 34 per cent (see Fig. 6) over other method by Buryachenko and Rammerstorfer (1997, 1998a, 1999b), based on hypothesis H_1 Eq. (4.10). Thus, the method of integral equations proposed enables one to discover the new effects and promises large benefits in analyses of a wide class of nonlinear problems for composite materials such as nonlinear elasticity, nonlinear viscosity and creeping, elastoplasticity, strength and fracture.

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Appendix A. Some relations for the Gaussian distribution

It is known that for Gaussian distribution of N-dimensional vector $\mathbf{e} = (e_1, \ldots, e_6) \in R^{N_6}$ ($N_6 = 6$), the probability density has the form

$$p(e_1, \ldots, e_N) = \frac{1}{\sqrt{(2\pi^N)\det(K^e)}} \exp\left(-\frac{1}{2} \sum_{k=1}^{N_6} \sum_{l=1}^{N_6} (e_k - \langle e_k \rangle)\left(K^e_{kl}\right)^{-1}(e_l - \langle e \rangle_l) \right), \tag{A1}$$

where $K^e_{kl} \equiv \langle (e_k - \langle e_k \rangle)(e_l - \langle e \rangle_l) \rangle$ is called the covariance matrix and $\det(K^e)$ is the determinant of the matrix K^e_{kl}. Then the higher correlations vanish and the moment of third, forth,... orders can be expressed by $\langle e_k \rangle$, Δe^2_{kl} ($k, l, m, n = 1, \ldots, N_6$)

$$\langle e_k e_l e_m \rangle = \langle e_k \rangle \langle e_l \rangle \langle e_m \rangle + \langle e \rangle_k \Delta e^2_{lm} + \langle e_l \rangle \Delta e^2_{km} + \langle e_m \rangle \Delta e^2_{kl},$$

$$\langle e_k e_l e_m e_n\rangle = \langle e_k\rangle\langle e_l\rangle\langle e_m\rangle\langle e_n\rangle + \Delta e^2_{kl}\Delta e^2_{mn} + \Delta e_{km}\Delta e^2_{ln} + \Delta e^2_{kn}\Delta e^2_{lm} + \langle e_k\rangle\langle e_l\rangle\Delta e^2_{mn} + \langle e_k\rangle\langle e_m\rangle\Delta e^2_{ln} + \langle e_m\rangle\langle e_n\rangle\Delta e^2_{kl} + \langle e_l\rangle\langle e_n\rangle\Delta e^2_{km} + \langle e_l\rangle\langle e_m\rangle\Delta e^2_{kn}. \tag{A2}$$

In particular one obtains $\langle e^3_k\rangle = \langle e_k\rangle^3 + 3\langle e_k\rangle\Delta e^2_{kk}$ and $\langle e^4_k\rangle = \langle e_k\rangle^4 + 3(\Delta e^2_{kk})^2 + 6\langle e_k\rangle^2\Delta e^2_{kk}$.

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